2A2C Introduction to Control Theory 3

Kostas Margellos

kostas.margellos@eng.ox.ac.uk

Michaelmas Term Term 2020

Note

With grateful acknowledgement to Mark Cannon for making material from Hilary Term 2017 available. Any remaining errors or typos should be referred to kostas.margellos@eng.ox.ac.uk

Questions

1. An aircraft autopilot uses the following continuous time PID controller to control roll angle:

$$u(t) = K\left(e(t) + \frac{1}{T_I}\int^t e(\tau)\,\mathrm{d}\tau + T_d\,\dot{e}(t)\right), \quad e(t) = r(t) - y(t)$$

   with gains $K = 2$, $T_d = 0.05$ and $T_i = 1$, where $u(t)$ is the control signal, $y(t)$ is the roll angle, $r(t)$ is a reference signal. The controller is to be implemented using a digital control system, hence it is discretized using a sampling interval T.

   (a) Using Euler's backward derivative approximation, provide an expression for the control input u_k at the k-th sampling instant in terms of the samples of e and u and the sampling interval T.

   (b) Determine the z-transform transfer function of the controller for $T = 0.5$.

2. If $F(z) = \mathcal{Z}\{f(kT)\} = \sum_{k=0}^{\infty} f(kT)z^{-k}$, show that:

   (a) $\mathcal{Z}\{f(kT - nT)\} = z^{-n}F(z)$, for $n > 0$

   (b) $\mathcal{Z}\{\alpha f_1(kT) + \beta f_2(kT)\} = \alpha F_1(z) + \beta F_2(z)$

(c) $\mathcal{Z}\{k\, f(kT)\} = -z\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{Z}\{f(kT)\}.$

3. State the signals that have the following Laplace transforms and then find from first principles their z-transforms after sampling with period T.

(a) $\frac{1}{s}$

(b) $\frac{a}{s(s+a)}$

(c) $\frac{2}{s^3}$

(d) $\frac{e^{-sh}}{s+a}$

Check your answer to (d) for $h = T$ and explain why it is discontinuous at $h = 0$.

Hint: In part (d) consider the sample corresponding to $k = 0$.

4. (a) Compute the z-transform of $x_k = a^k$, where $k \geq 0$, $a > 0$.

(b) Compute the z-transform of $x_k = \cos ak$, where $k \geq 0$, and a is an arbitrary constant.

(c) Compute the inverse z-transform of $X(z) = \frac{8z-19}{(z-2)(z-3)}$.

(d) Compute the inverse z-transform of $X(z) = \frac{0.2z^{-1}}{1-0.5z^{-1}}$.

5. A digital-to-analogue converter (DAC) is updated every T seconds. The DAC is connected via a zero-order hold to a current amplifier with a gain k_A A/V which in turn drives a motor with inertia J kg m^2 and torque constant k_T N m/A. A tacho-generator with a gain of k_V V/(rad s^{-1}) is sampled through an ADC synchronously with the DAC. Calculate the pulse transfer function $G(z)$ of the system seen by the computer.

6. The computer in Question 5 takes a time τ to calculate its control action and the DAC is therefore updated a time τ after the ADC sampling instant, where $0 < \tau < T$. Calculate the z-transform of the continuous time system with this extra delay incorporated. Check your answer for the cases $\tau = T$ and as $\tau \to 0$.

7. Show that the mapping $z = e^{sT}$ maps strips in the left half of the s-plane into the unit circle in the z-plane. Find the locus of $z = e^{sT}$ in the z-plane when $s = \sigma + j\omega$ and:

   (a) σ is constant;

   (b) ω is constant.

8. (a) Derive expressions for the images in the z-plane of the s-plane poles with natural frequency ω_0 and damping ratio ζ, under the mapping $z = e^{sT}$.

   (b) Determine the damping ratio and natural frequency of the z-plane poles at $z = \frac{1}{4} \pm j\frac{\sqrt{3}}{4}$ if the sampling interval is T.

   (c) Consider a closed loop system described by the transfer function

$$\frac{Y(z)}{R(z)} = \frac{3z}{z^2 - 0.5z + 0.25}.$$

   Compute the steady state value and the settling time of its response to a unit step. The sampling interval is $T = 1$ second.

9. (a) Consider the continuous time linear system

$$\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t)$$
$$y(t) = \bar{C}x(t) + \bar{D}u(t),$$

   where

$$\bar{A} = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Given a fixed sampling period T, compute the matrices (A, B, C, D) of the state space description of the associated sampled data linear system.

(b) Consider a discrete time linear system in state space form, whose A matrix is given by

$$A = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}.$$

Compute the *zero input transition* of the system's solution when the initial state is $x_0 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

10. Consider a continuous time linear system governed by the differential equation

$$\dot{x}(t) = Ax(t),$$

where $A \in \mathbb{R}^{n \times n}$ is assumed to be diaognalizable. Using Euler's forward approximation to first order derivatives, we have that $\dot{x}(t) \approx \frac{x_{k+1} - x_k}{T}$, where T is the time-step size. This suggests the following discrete time approximation

$$\frac{x_{k+1} - x_k}{T} = Ax_k \;\Leftrightarrow\; x_{k+1} = (I + AT)x_k,$$

where I is an $n \times n$ identity matrix.

(a) Show that for all $i = 1, \ldots, n$, if λ_i is an eigenvalue of A associated with the eigenvector w_i, then $1 + \lambda_i T$ is an eigenvalue of $I + AT$ associated with the same eigenvector.

(b) Show that the solution of the system is given by

$$x_k = W(I + \Lambda T)^k W^{-1} x_0,$$

where W is a matrix whose columns are the eigenvectors of A, Λ is a diagonal matrix whose diagonal entries are the eigenvalues of A, and x_0 is the initial state.

(c) Provide a condition for T such that the discrete time approximation is asymptotically stable.

Answers to selected questions

1. (a). Using backwards differences to approximate derivatives:

$$u_k = K\left(1 + \frac{T_d}{T} + \frac{T}{T_i}\right)e_k - K\left(1 + 2\frac{T_d}{T}\right)e_{k-1} + K\frac{T_d}{T}e_{k-2} + u_{k-1}$$

   (b). $\frac{U(z)}{E(z)} = \frac{3.2z^2 - 2.4z + 0.2}{z(z-1)}$ if $T = 0.5$ sec

3. (a). $\frac{z}{z-1}$
   (b). $z\frac{(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
   (c). $T^2\frac{z(z+1)}{(z-1)^3}$
   (d). $\frac{e^{-a(T-h)}}{z-e^{-aT}}$

4. (a). $\frac{z}{z-a}$
   (b). $\frac{z(z-\cos a)}{z^2 - 2z\cos a + 1}$
   (c). $-\frac{19}{6}\delta_k + \left(\frac{3}{2}2^k + \frac{5}{3}3^k\right)\mathcal{U}_k$
   (d). $0, 0.2, 0.1, 0.05, \ldots$

5. $\frac{k_A k_T k_V T}{J(z-1)}$

6. $\frac{k_A k_T k_V (Tz - \tau z + \tau)}{Jz(z-1)}$

8. (a). $z = re^{j\theta}$ with $r = e^{-\zeta\omega_0 T}$, $\theta = \omega_0 T\sqrt{1-\zeta^2}$
   (b). $\omega_0 = \frac{1.256}{T}$, $\zeta = 0.552$.
   (c). Steady state value: $y_{ss} = 4$; Settling time: $T_s = 6.63$ seconds.

9. (a). $A = \begin{bmatrix} 2e^{-T} - e^{-2T} & 2e^{-T} - 2e^{-2T} \\ -e^{-T} + e^{-2T} & -e^{-T} + 2e^{-2T} \end{bmatrix}$.
   (b). $x_k = (-1)^{k+1}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.