University of Oxford
Department of Engineering Science
Engineering Science 3rd Year

# B15 Control Systems Laboratory

## Hilary Term 2025

### Lab Organisers



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### Location

Control Laboratory
Thom Building 5th Floor

### Safety Statement

You are reminded to observe safe laboratory working practices in accordance with the course handbook: Canvas: MEng Engineering Science: Handbooks: FHS Handbook 24-25.pdf

**Acknoeledgements:** Acknowledgements to Professor Mark Cannon, for making an earlier version of the lab material available.

# Coupled Tanks Experiment

## Instructions

- Read these instructions, carry out the preparatory work (Questions 1-7) writing your answers in the answer sheet, and upload these to the preparation submission page on Canvas before you start the lab. Your attendance will not be registered unless the lab demonstrator is satisfied that you have completed the preparatory work.
- The experimental part of the lab should take no more than about 5 hours in total. Demonstrators will be available 11:00-17:00 on the day of your scheduled lab session. Register for a lab session using the lab signup page. Students are expected to work in pairs.
- During the lab, answer the remaining questions, namely, Questions 8-30 by filling in the answer sheet and submit the completed answer sheet to the report submission page on Canvas. A lab demonstrator will discuss your work with you and assign you a mark for the lab. Each student must provide a complete set of answers.

## Overview of Experiment

The laboratory exercises explore the design of controllers for water levels in a pair of connected tanks using a pump and level sensor measurements. The lab makes use of the analysis techniques and design tools introduced in the B15 Linear Dynamic Systems course (linearisation, controllability, observability) and the B15 Optimal Control course (linear-quadratic regulation, integral action, state estimation). You may find it helpful to refer to the B15 lecture notes while you work through the exercises.

The experiment consists of three phases:

1. Preparatory work (which must be completed before the lab): develop a linear model of the process and relate its parameters to system properties.

2. Familiarisation with the experiment: perform tests to identify the parameters of a state space model of the coupled tanks system.

3. Design and implement controllers: linear-quadratic optimal control; implementation of integral action; state observer design and implementation.

## Safety

The laboratory equipment uses a pump to transfer water into two tanks. Before you begin, please ensure that all connecting pipes are secured to the apparatus (ask a demonstrator if you need help), and familiarise yourself with the equipment, including the location of the kill switch for the pump. Please also ensure that all electrical equipment (power supply, amplifier, computer) is kept free of water during the lab. A risk assessment is available in the laboratory.

## Learning Outcomes

1. Ability to create a linearised state space model of a nonlinear system.

2. Identify the parameters of the model and validate these using the response of the experimental apparatus.

3. Design and implement a linear-quadratic optimal controller.

4. Understand controller limitations and the significance of cost weights.

5. Design and implement a linear-quadratic optimal controller with integral action to eliminate steady state error.

6. Design a state observer.

7. Design and implement a linear-quadratic optimal output feedback controller using a state observer

# 1 Introduction

The coupled tanks experiment is shown schematically in Figure 1. A voltage $v$ drives an electric motor which pumps water from a reservoir into Tank 1. The flow rate through the pump is $q_{\text{in}}$, and this can be assumed to be proportional to $v$. Water from Tank 1 flows into Tank 2 through an orifice, then returns to the reservoir via a second orifice.

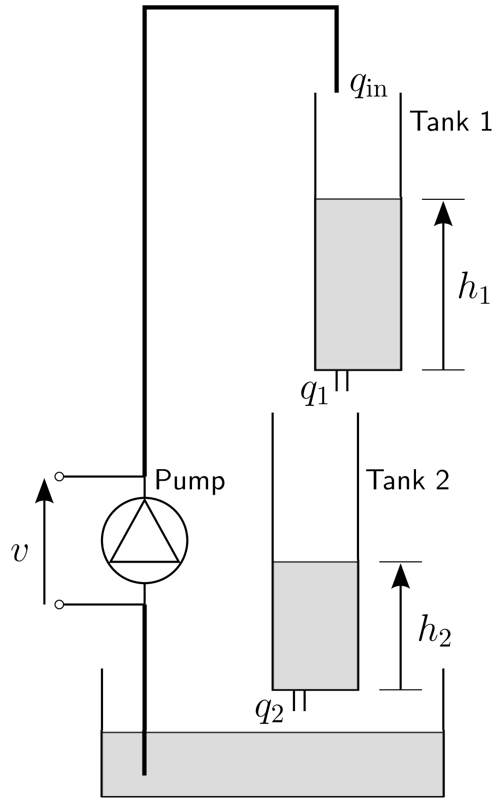


**Figure 1:** Schematic of coupled tanks experiment

The objective is to control the water level in Tank 2 to a specified target level (reference value or setpoint). This is analogous to the operation of holding tanks in a processing plant, which are used to ensure a consistent and reliable flow of material between the individual stages of a multi-stage process.

In order to implement feedback control on the system, a pair of level sensors provide voltages which are proportional to the tank levels $h_1$ and $h_2$. These signals are filtered to reduce high frequency noise and converted to give the values of the water levels in cm. The setpoint for $h_2$ is assumed to be subject to arbitrary step changes determined by a human operator, but otherwise is assumed to be constant.

For analysis purposes it is convenient to make the following assumptions:

1. Sensor and actuator dynamics can be ignored, i.e. they are much faster than the coupled tank system dynamics;

2. The sampling rate is sufficiently high that continuous time system models and continuous time control design techniques can be used, i.e. the control action as perceived by the coupled tanks system is effectively the same as that provided by a continuous-time controller.

## 2 Preparatory Work: Modelling

Assume that the pump flow rate $q_{\text{in}}$ depends linearly on the pump voltage $v$:

$$q_{\text{in}} = k_p v$$

for some fixed gain $k_p$. The equations for (volumetric) flow-rates $q_1$ and $q_2$ out of tanks 1 and 2 are

$$q_1 = \sigma_1 a_1 \sqrt{2gh_1} \tag{2.1}$$
$$q_2 = \sigma_2 a_2 \sqrt{2gh_2} \tag{2.2}$$

where the levels $h_1$ and $h_2$ are measured relative to the bottoms of tank 1 and tank 2, respectively, $\sigma_1$, $\sigma_2$ are orifice constants, $a_1$, $a_2$ are cross-sectional areas of the orifices, and $g$ is the acceleration due to gravity. If the cross-sectional area of each tank is $A$, then the differential equations relating the heights of the water in the two tanks to the flow rates are:

$$A\,\dot{h}_1 = k_p v - q_1 \tag{2.3}$$
$$A\,\dot{h}_2 = q_1 - q_2 \tag{2.4}$$

where $\dot{h}_i(t) = dh_i/dt$ is the derivative of $h_i(t)$ with respect to time, $t$. These equations are nonlinear due to the square roots appearing in the expressions for $q_1$ and $q_2$ but they can be linearised about an equilibrium point.

Question 1. Provide expressions for the equilibrium values, $h_1^0$, $h_2^0$, of the levels, $h_1$, $h_2$, in the tanks when the pump voltage $v$ is held at $v = v^0$, where $v^0$ is a constant.

Question 2. Let $h_1 = h_1^0 + x_1$ and $h_2 = h_2^0 + x_2$, where $x_1$ and $x_2$ are small deviations from the equilibrium levels $h_1^0$, $h_2^0$, respectively. Find the Taylor series expansions of $q_1$ in (2.1) about $h_1 = h_1^0$ and $q_2$ in (2.2) about $h_2 = h_2^0$, truncated after the linear terms in $x_1 = h_1 - h_1^0$ and $x_2 = h_2 - h_2^0$.

[Hint: An easier way to linearize the system is by exploiting the Taylor expansion $(1 + x_i)^{1/2} = 1 + \frac{1}{2}x_i - \frac{1}{8}x_i^2 + \cdots$, for $i = 1, 2$. For each $i = 1, 2$, the resulting approximations will be of the form $q_i \approx \alpha_i + \beta_i x_i$ for constants $\alpha_i$, $\beta_i$ to be determined.]

Now let $v = v^0 + u$, where $u$ is a small deviation from the equilibrium pump voltage $v^0$. Use equations (2.3)-(2.4) and the linearised flow-rate expressions in Question 2 to create the linearised state-space model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1/T_1 & 0 \\ 1/T_1 & -1/T_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u, \tag{2.5}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{2.6}$$

where the output $y$ is the deviation of $h_2$, the height of the fluid in the second tank, from $h_2^0$.

Question 3. Provide expressions for the model parameters $T_1$, $T_2$, $b_1$ and $b_2$.

Hence show that

$$T_1 = \frac{2Ah_1^0}{k_p v^0}, \quad T_2 = \frac{2Ah_2^0}{k_p v^0}. \tag{2.7}$$

Question 4. Use the state space model (2.5)-(2.6) to determine the transfer functions from $U(s)$ to $X_1(s)$ and from $U(s)$ to $X_2(s)$.

Question 5. Write down expressions for the dc gain and time-constant of $G_1(s)$.

Question 6. Write down expressions for the dc gain, the undamped natural frequency and the damping ratio of $G_2(s)$.

Question 7. Determine the $2\%$ settling times of the step responses of $G_1(s)$ and $G_2(s)$, i.e. the time taken for the step responses of $G_1(s)$ and $G_2(s)$ to reach $2\%$ of their steady state values.

[Hint: For a first order system with time constant $T$ the $2\%$ settling time is

$$T_{s1} \approx 4T,$$

since $e^{-4} \approx 0.02$. For a second order system with damping ratio $\zeta < 1$ and undamped natural frequency $\omega_0$ the $2\%$ settling time is

$$T_{s2} \approx 4/\zeta\omega_0,$$

since the time constant of the exponentially decaying oscillation in the step response is $1/\zeta\omega_0$. Note that for our system it turns out that $\zeta \approx 1$; however, using the same approximation would still give a reasonable estimate for the settling time.]

# 3 Experiment

The modelling and controller design parts of this laboratory will be performed using Matlab with controllers implemented using Simulink. The Matlab Coder is used to automatically generate code from Simulink models. This provides the necessary instructions to read in data from the water level sensors, compute the control action and output control signals to the pump.

Matlab should be running on the laboratory PC when you begin the exercise, but if it is not, or if you need to restart the computer, log in using

username: `lab5`    password: `access`

The files for the lab are in the folder `C:\work\CONTROL\B15lab`. You can change these files as you work through the exercises (they will be erased before the next lab session). You will find it helpful to write Matlab scripts to solve the controller design problems and answer the questions below.

- To start the first part of the experiment click on `b15_lab_a.slx` or type `b15_lab_a` at the Matlab prompt to open the window shown in Figure 2. *Note that it usually takes a few seconds to load the model.*

- Build and run the Simulink model (i.e. generate the code, compile it and run it) by clicking the green "Run" button on the toolbar indicated in Figure 2. *Note that it will take a few seconds to build the model.*

- Change the voltage input to the pump by opening the `Controller` subsystem, clicking on the box labelled "Manual pump demand, v0" and changing the value the gain from the default value of $7$.

- Open the scopes `h1_response` and `h2_response`; these provide plots of the water levels $h_1$ and $h_2$ in cm.

## 3.1 Parameter Identification

For most of the experiment the set-point for $h_2$ will be around $14$-$16$ cm. It is therefore reasonable to consider the linearization you have obtained for (2.3)-

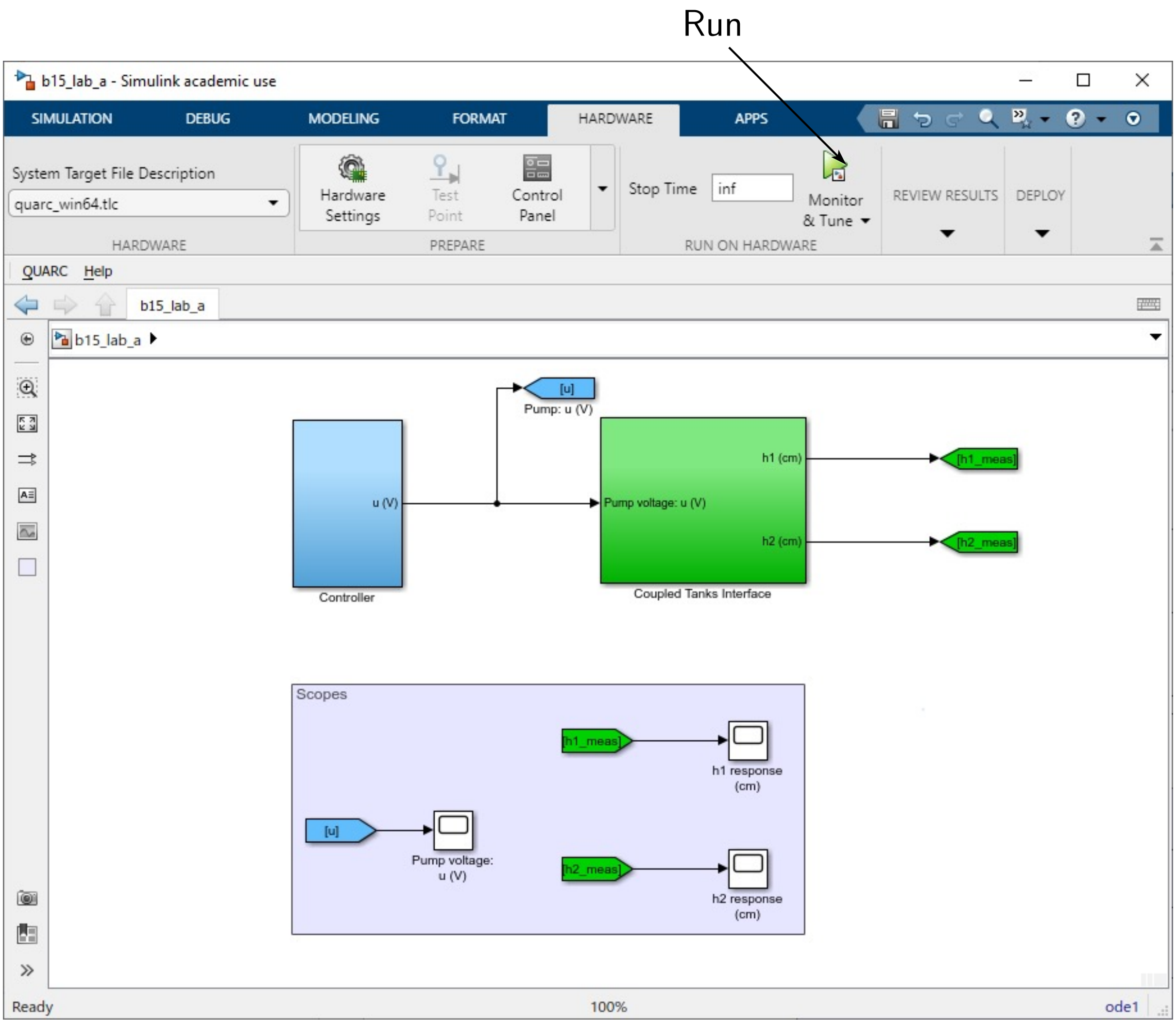


**Figure 2:** Simulink model `b15_lab_a`

(2.4) about an equilibrium level of $h_2^0 \approx 15\,\text{cm}$. This is typically obtained with a pump voltage $v^0$ of about 7-8 V. Choose a suitable value for $v^0$ and record the corresponding steady state values, $h_1^0$ and $h_2^0$. Note that $h_2^0$ does not need to be exactly equal to $15\,\text{cm}$, we just need $14 \leq h_2^0 \leq 16$, and that takes several minutes to reach steady state.

Each of the tanks is cylindrical with internal diameter $4.4\,\text{cm}$ and the pump gain is $k_p = 3.3\ \text{cm}^3\,\text{s}^{-1}\,\text{V}^{-1}$.

Question 8. Determine numerical values for $b_1$, $b_2$, $T_1$ and $T_2$ using $h_1^0$, $h_2^0$, $v^0$ and (2.7).

Question 9. Use the expressions derived in Section 2 and your values for $b_1$, $b_2$, $T_1$ and $T_2$ to determine the d.c. gain and time constant of $G_1(s)$

Question 10. Determine the d.c. gain, the undamped natural frequency $\omega_0$ and the damping ratio $\zeta$ of $G_2(s)$

Question 11. Estimate the $2\%$ settling times of the step responses of $G_1(s)$ and $G_2(s)$.

## 3.2 Model Validation

Apply a small change (e.g. $\pm 0.2\,$V) to $v$ to generate a new equilibrium point for $h_1$ and $h_2$ in the range $14$-$16\,$cm.

Question 12. Record the new value of $v$ and new steady state levels in tanks 1 and 2 and use these together with your values for $v^0$, $h_1^0$ and $h_2^0$, to estimate experimental values for the d.c. gains of $G_1(s)$ and $G_2(s)$.

Now change the value of $v$ back to $v^0$ and record the resulting open loop step responses. This can be done by copying the plots shown in the scopes (e.g. using "Copy to Clipboard") and pasting into a Word document.

Question 13. Estimate the settling times of the responses of the levels in tanks 1 and 2 from the open loop step responses.

Question 14. Compare the experimental values for the d.c. gains of $G_1(s)$ and $G_2(s)$ with the values that you predicted in Section 3.1.

## 3.3 Linear Quadratic Regulator (LQR)

The state space model of the linearised system (2.5)-(2.6) can be expressed

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t), \tag{3.1}$$

$$y(t) = C\mathbf{x}(t), \tag{3.2}$$

where $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $A = \begin{bmatrix} -1/T_1 & 0 \\ 1/T_1 & -1/T_2 \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$.

We consider an infinite horizon linear quadratic regulation (LQR) optimal control problem, where the control law $u(t)$ that minimizes the cost function:

$$\int_0^\infty \left[\mathbf{x}^\top(t) Q \mathbf{x}(t) + u^\top(t) R u(t)\right] \mathrm{d}t,$$

evaluated for the model (3.1)-(3.2) and for given weighting matrices $Q$, $R$, is

$$u(t) = -K\mathbf{x}(t), \text{ where } K = \begin{bmatrix} k_1 & k_2 \end{bmatrix},$$

where $K$ is the optimal LQR feedback gain matrix. This set of gains will be determined here numerically by solving the associated steady-state Riccati equation (see B15 lectures) in Matlab, via the command

$$K = \texttt{lqr}(A, B, Q, R).$$

In this experiment we are interested in controlling the level in tank 2, and we therefore wish to penalize the output $y(t) = x_2(t)$ in the cost. This is achieved by choosing $Q = C^\top C$ so that $\mathbf{x}(t)^\top Q \mathbf{x}(t) = y^2(t)$.

For $R = 0.1, 0.05, 0.01$, compute:

- the optimal gain matrix $K$;
- the closed loop poles $p_{\text{cl}}$ (i.e. the eigenvalues of $A - BK$);
- their undamped natural frequency $\omega_0$ and damping ratio $\zeta$;
- the $2\%$ settling time, $T_s$, of the step response.

Question 15. Record the values of $R$, $K$, $p_{\text{cl}}$, $\omega_0$, $\zeta$ and $T_s$ in a table.

Question 16. Describe the effects of increasing $R$ and explain why $R < 0.01$ is likely to result in poor performance.

Question 17. Choose a suitable value for $R$ using your answer to Question 16.

Open the Simulink model `b15_lab_b.slx` shown in Figure 3, open the block marked `Controller` and insert the value of $v^0$ (from Section 3.1) and the gains $k_1$ and $k_2$ corresponding to the chosen $R$. Run the controller and test the step response of the closed loop system by changing the setpoint $h_{2,\text{set}}$ between the values of $14\,\text{cm}$ and $16\,\text{cm}$.

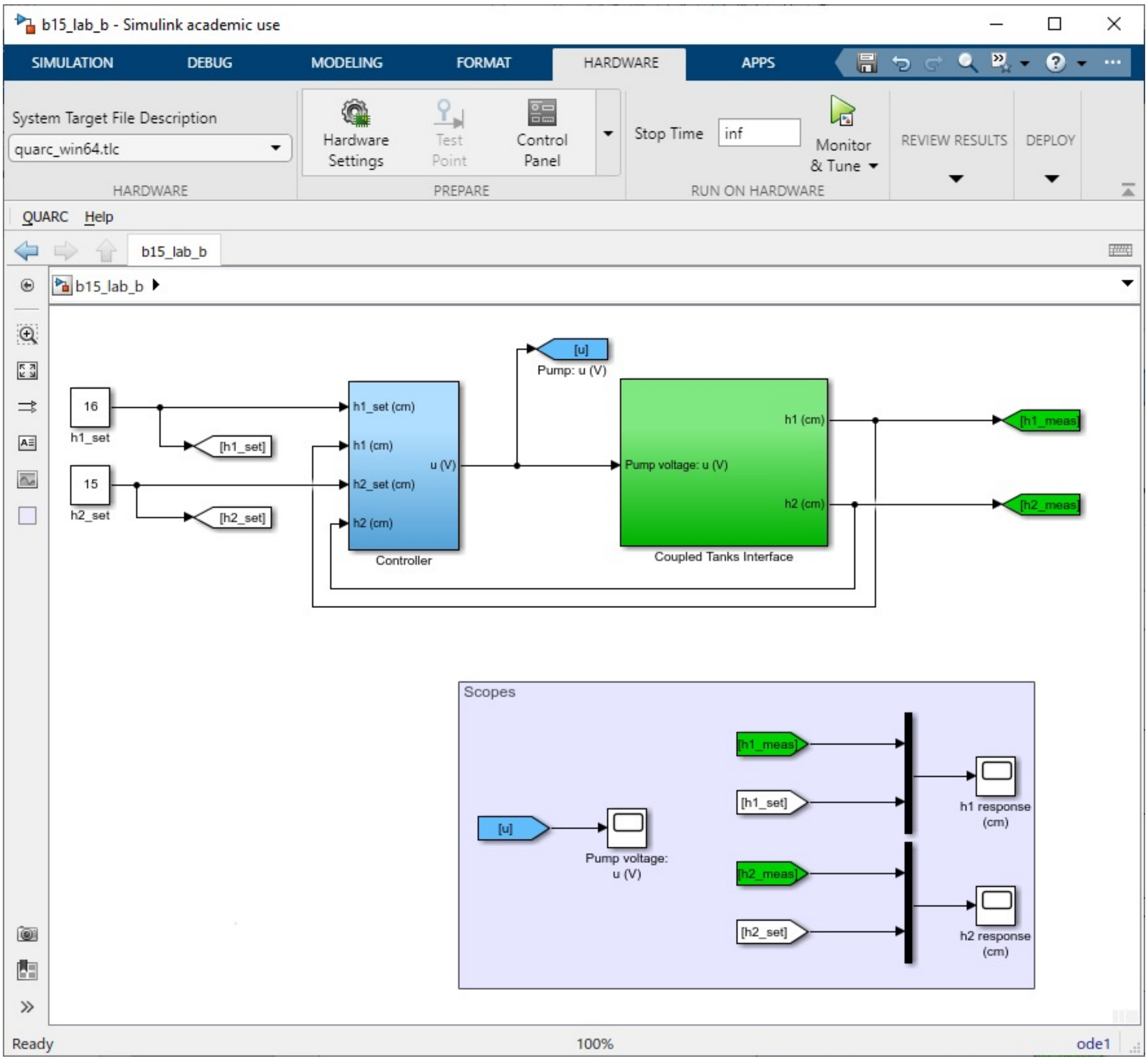


**Figure 3:** LQ optimal controller implementation in Simulink model `b15_lab_b`

Question 18. Record the step responses and comment on the accuracy of the predictions of $\zeta$ and $T_s$.

Question 19. Explain what causes the steady state errors in $h_2$ and $h_1$.

## 3.4 LQR with Integral Action

To force the controller to include a term that depends on the integral of the error in $h_2$, we augment the system model so that the integral of the error between $h_2$ and its setpoint appears as a state. To this end, denote the integral of this error as

$$x_3(t) = \int_0^t y(\tau)d\tau = \int_0^t x_2(\tau)d\tau \implies \dot{x}_3(t) = y(t) = x_2(t).$$

Notice that the tracking error $y = x_2 = h_2 - h_2^0$, where $h_2^0$ is taken to be the setpoint of $h_2$. We can incorporate this new differential equation in the state space system description of the previous section. We will this obtain an augmented state vector $\mathbf{x}_a^\top(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}$ and the associated augmented state space description

$$\dot{\mathbf{x}}_a(t) = A_a\mathbf{x}_a(t) + B_a u(t), \tag{3.3}$$

$$y(t) = C_a\mathbf{x}_a(t), \tag{3.4}$$

where $A_a = \begin{bmatrix} -1/T_1 & 0 & 0 \\ 1/T_1 & -1/T_2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B_a = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}$, $C_a = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$.

The infinite horizon LQR problem for the augmented system takes the form

$$\int_0^\infty \left[\mathbf{x}_a^\top(t)Q_a\mathbf{x}_a(t) + u^\top(t)Ru(t)\right] \mathrm{d}t, \text{ where } Q_a = \begin{bmatrix} C^\top C & 0 \\ 0 & q \end{bmatrix}, \quad q > 0.$$

Here $\mathbf{x}_a(t)^\top Q_a\mathbf{x}_a(t) = y^2(t) + qx_3^2(t)$, so the weight $q$ allows the designer to specify the penalty on the integral of the tracking error, which now appears in the cost due to the term $qx_3^2(t)$. Notice that the optimal LQR controller now takes the form

$$\begin{aligned} u(t) &= -K_a\mathbf{x}_a(t), \\ &= -\begin{bmatrix} k_{a,1} & k_{a,2} \end{bmatrix} \mathbf{x}(t) - k_{a,3}\int_0^t y(\tau)d\tau, \end{aligned}$$

where $K_a = \begin{bmatrix} k_{a,1} & k_{a,2} & k_{a,3} \end{bmatrix}$. It now becomes prominent that the controller includes an integral action.

With the value of $R$ that was chosen in question 17, use the Matlab function `lqr` to compute the optimal controller $K_a$ for $q = 0.1, 0.05, 0.01$. In each case, compute

- the optimal gain matrix $K_a$;
- the closed loop poles $p_{\text{cl}}$ (i.e. the eigenvalues of $A_a - B_a K_a$);
- the undamped natural frequency $\omega_0$ and damping ratio $\zeta$ of any complex poles;
- the $2\%$ settling time, $T_s$ corresponding to the slowest closed loop pole.

Question 20. Record $q$, $K_a$, $p_{\text{cl}}$, $\omega_0$, $\zeta$, $T_s$ in a table.

Question 21. Describe the effects of increasing the weight $q$ and explain why $q > 0.1$ is likely to give poor performance.

Question 22. Choose a suitable value for $q$.

Open the Simulink model `b15_lab_c.slx`. Open the `Controller` block and insert the value of $v^0$ (from Section 3.1) and the gains $k_{a,1}$, $k_{a,2}$ and $k_{a,3}$ that are obtained with your chosen value of $q$. Run the controller and test the step response of the closed loop system for step changes in $h_{2,\text{set}}$ between $14\,\text{cm}$ and $16\,\text{cm}$.

Question 23. Record the step response and compare the step response obtained in Question 18.

## 3.5 State Estimation

The feedback laws designed in Sections 3.3 and 3.4 were computed assuming that the full state of the system can be measured. This section assumes that only the level of tank 2, $y(t)$, can be measured, and is hence the only state that can be used for control. To this end, an observer is used to estimate the level of tank 1 (as shown in Fig. 4) for use in a feedback control law. Defining the state estimate constructed by a linear observer by $\hat{\mathbf{x}}(t) = \begin{bmatrix} \hat{x}_1(t) & \hat{x}_2(t) \end{bmatrix}^\top$. Its evolution is governed by the differential equation

$$\dot{\hat{\mathbf{x}}}(t) = (A - LC)\hat{\mathbf{x}}(t) + Bu(t) + Ly(t). \tag{3.5}$$

where the observer gain, $L$, is to be chosen by the designer. Recalling that $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t)$ and $y(t) = C\mathbf{x}(t)$, we can construct the estimation error $e(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$. Its evolution is given by

$$\dot{e}(t) = (A - LC)e(t). \tag{3.6}$$

The observer gain $L$ can be computed so that the estimation error converges to zero (in magnitude). This can be achieves by appropriately placing the eigenvalues of $A - LC$ using the Matlab function `place`:

$$L = \texttt{place}(A', C', p_{\text{obs}})',$$

where $p_{\text{obs}} = \begin{bmatrix} p_{\text{obs},1} & p_{\text{obs},2} \end{bmatrix}$, and $p_{\text{obs},1}$, $p_{\text{obs},2}$ are the desired eigenvalue/pole locations.

Question 24. Explain why $p_{\text{obs},1}$, $p_{\text{obs},2}$ must have negative real parts.

The level sensor for $h_2$ is subject to noise at frequencies above $3\,\text{rad}\,\text{s}^{-1}$.

Question 25. Explain why it is desirable to make $|\text{Re}[p_{\text{obs}}]|$ an order of magnitude less than $3$. By $|\cdot|$ we mean the absolute value of its argument.

Question 26. By referring to the closed loop poles $p_{\text{cl}}$ computed in Section 3.4, suggest a lower bound on the value of $|\text{Re}[p_{\text{obs}}]|$. [Hint: Consider what determines the dominant poles of the closed loop system.]

Question 27. Choose suitable locations for $p_{\text{obs}}$ and compute the gain $L$.

Open the Simulink model `b15_lab_d.slx` shown in Figure 4. Open the block marked `Observer` and enter the values of $v^0$, $h_1^0$ and $h_2^0$ in the required boxes (alternatively, just ensure that $v^0$, $h_1^0$ and $h_2^0$ are correctly defined in the Matlab workspace). Ensure that that the correct model is entered in the block marked `State-Space` (the easiest way to do this is to define $A$, $B$, $L$, $C$ in the Matlab workspace). Insert the gains computed in Section 3.4 with the chosen values for $R$ and $q$ into the `Controller` block, run the controller and test the step response of the closed loop system for changes in $h_{2,\text{set}}$ between $14\,\text{cm}$ and $16\,\text{cm}$.

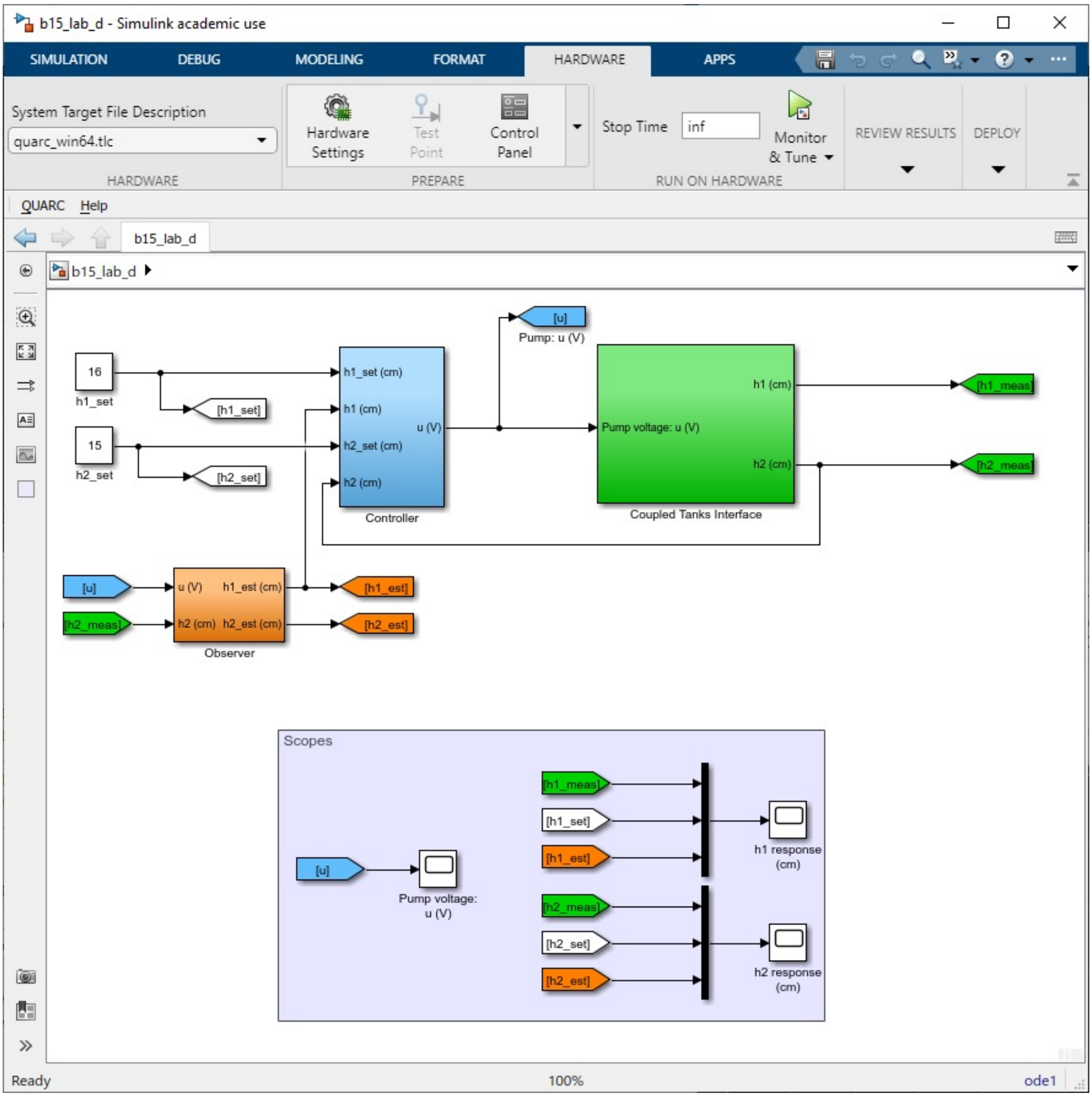


**Figure 4:** LQ optimal controller with state estimation using an observer in Simulink model `b15_lab_d`

Question 28. Record the step response and comment on the effects of using an observer to estimate $h_1$.

Question 29. Comment on the accuracy of the estimates of $h_1$ and $h_2$.

A simplified block diagram of the closed loop system, including sensor noise, $n$, is shown in Fig. 5. Here we consider the augmented system where we account for the integral of the tracking error.

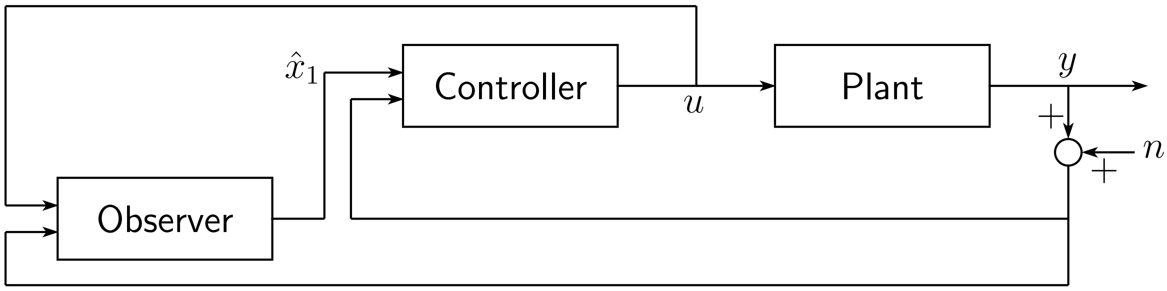


**Figure 5:** Closed loop system block diagram

Question 30. Record the Bode diagrams for the transfer functions from $N(s)$ to $Y(s)$ and from $N(s)$ to $\hat{X}_1(s)$. Does the chosen value of $L$ satisfy the requirements that were used to determine upper and lower bounds on the observer poles $p_{\text{obs}}$?

[Hint: Since now the output is affected by a noise signal $n$, we need to replace all terms that depend on $y$ with $y + n$ in the equations of the augmented system and the observer. To this end, consider (3.3) and (3.5) (we drop the dependency of the variables on $t$ for simplicity); upon this replacement they can be written as

$$\dot{\mathbf{x}}_a = A_a \mathbf{x}_a + B_a u + B_n n, \tag{3.7}$$

$$\dot{\hat{\mathbf{x}}} = (A - LC)\hat{\mathbf{x}} + Bu + L(y(t) + n), \tag{3.8}$$

where $B_n = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$. The term $B_n n$ appears in (3.7) since the last component of the augmented state will now involve the integral of $y + n$ (as the output is affected by noise). This is the third state in $\mathbf{x}_a$, hence the third entry of $B_n$ is one.

Recall now that $K_a = \begin{bmatrix} k_{a,1} & k_{a,2} & k_{a,3} \end{bmatrix}$. Let $K^{(1)} = \begin{bmatrix} k_{a,1} & 0 \end{bmatrix}$ and $K^{(2)} = \begin{bmatrix} 0 & k_{a,2} & k_{a,3} \end{bmatrix}$. The control law including the sensor noise signal $n$ can be then expressed as

$$u = -K^{(1)}\hat{\mathbf{x}} - K^{(2)}\mathbf{x}_a - k_{a,2} n, \tag{3.9}$$

where we made explicit the dependency on the state $\mathbf{x}_a$, the estimated state $\hat{\mathbf{x}}$, and the noise signal $n$. Notice that $K^{(1)}\hat{\mathbf{x}} = k_{a_1}\hat{x}_1$, as we employ the observer to use an estimate for the state we do not measure directly. Moreover, we have the term involving $n$ as the control gain $k_{a,2}$ would be multiplied with the second state that is available via the output as sensor measurement. Since the output is affected by noise, we apply this gain on $x_2 + n = y + n$ instead.

By combining equations (3.7), (3.7), and (3.9), the closed loop system model becomes

$$\begin{bmatrix} \dot{\mathbf{x}}_a \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} A_a - B_a K^{(2)} & -B_a K^{(1)} \\ LC_a - BK^{(2)} & A - LC - BK^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_a \\ \hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} B_n - B_a k_{a,2} \\ L - Bk_{a,2} \end{bmatrix} n,$$

$$\begin{bmatrix} y \\ \hat{x}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_a \\ \hat{\mathbf{x}} \end{bmatrix}.$$

Notice that $\hat{x}_1$ constitutes a virtual output as we employ for control and is not available in the actual output (sensor measurement). The Bode plot of the transfer function (matrix)

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

of a system defined by $\dot{x} = Ax + Bu$ and $y = Cx + Du$ can be obtained using the Matlab commands $\texttt{sys} = \texttt{ss}(A, B, C, D)$ and $\texttt{bode}(\texttt{sys})$.]