B15 Linear Dynamic Systems and Optimal Control
Example Paper 1

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Questions

1. Consider the amplifier of Figure 1, where $v_i(t)$ is the input voltage and $v_o(t)$ the output one. Denote by $v_{C_1}(t)$ and $v_{C_2}(t)$ the voltage across the capacitor C_1 and C_2, respectively, and assume that the amplifier is ideal.

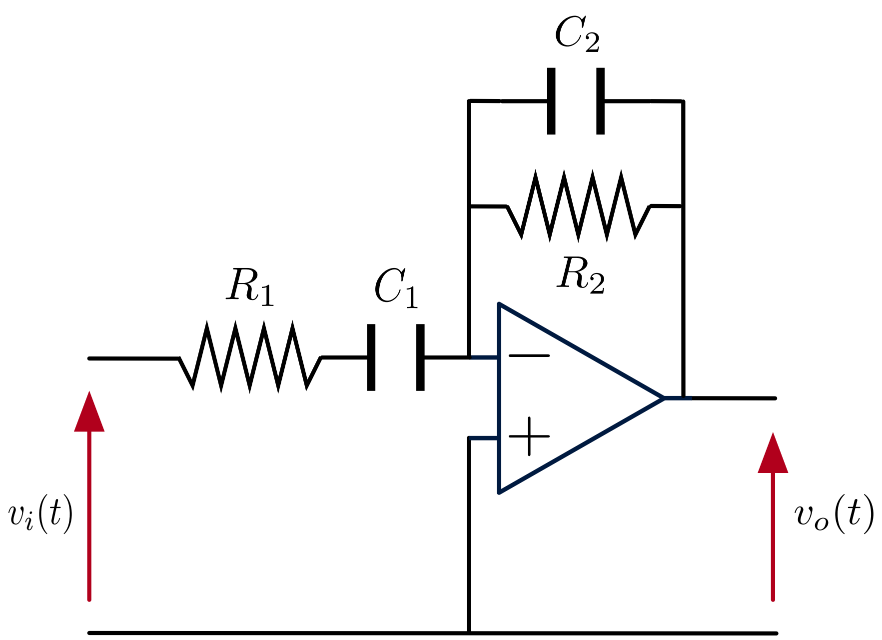

Figure 1: Amplifier circuit.

   (a) Derive the ordinary differential equations (ODEs) that capture the evolution of $v_{C_1}(t)$ and $v_{C_2}(t)$.

   (b) Provide a state space description of the amplifier circuit. What is the order of the resulting system?

   (c) Is the resulting system autonomous? Is it linear? Justify your answers.

2. Consider the following dynamical system

$$\ddot{z}(t) = 1 - \frac{1}{(z(t) + z^\star)^2} u(t),$$

where $z^\star > 0$ is a fixed parameter.

(a) Find the constant input $u(t) = u^\star$ for all $t \geq 0$ that renders 0 an equilibrium of the system, i.e., starting at $z(t) = 0$ the system does not move.

(b) Use $x(t) = \begin{bmatrix} z(t) & \dot{z}(t) \end{bmatrix}^\top$ and $y(t) = z(t)$ as the state vector and system output, respectively. Write the given dynamical system in state space form. Is the resulting system linear?

(c) Linearize the system around $x^\star = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$ and the value for $u^\star$ computed in part (a).

3. Recall that the state transition matrix for linear time invariant (LTI) systems with starting time zero is given by

$$\Phi(t) = e^{At} = I + At + \frac{A^2 t^2}{2!} + \ldots + \frac{A^k t^k}{k!} + \ldots$$

Show that it satisfies the following properties:

(a) $\Phi(0) = I$.

(b) $\frac{d}{dt}\Phi(t) = A\Phi(t)$.

(c) $\Phi(t)\Phi(-t) = \Phi(-t)\Phi(t) = I$.
What is the role of $\Phi(-t)$ in this case?

(d) For any $t_1, t_2 \in \mathbb{R}$, $\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2)$.

4. For each case below comment on whether matrix A is diagonalizable, and determine the matrix exponential e^{At}.

(a) $A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$, where $\omega \neq 0$.

(b) $A = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix}$, where $\sigma \neq 0$.

(c) $A = \begin{bmatrix} \lambda_1 & \lambda_2 - \lambda_1 \\ 0 & \lambda_2 \end{bmatrix}$, where $\lambda_1,\ \lambda_2 \neq 0$.

(d) $A = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

5. The so called Wien oscillator is the circuit shown in Figure 2 with $k > 1$. Denote by $v_{C_1}(t)$ and $v_{C_2}(t)$ the voltage across the capacitor C_1 and C_2, respectively, and assume that the amplifier is ideal.

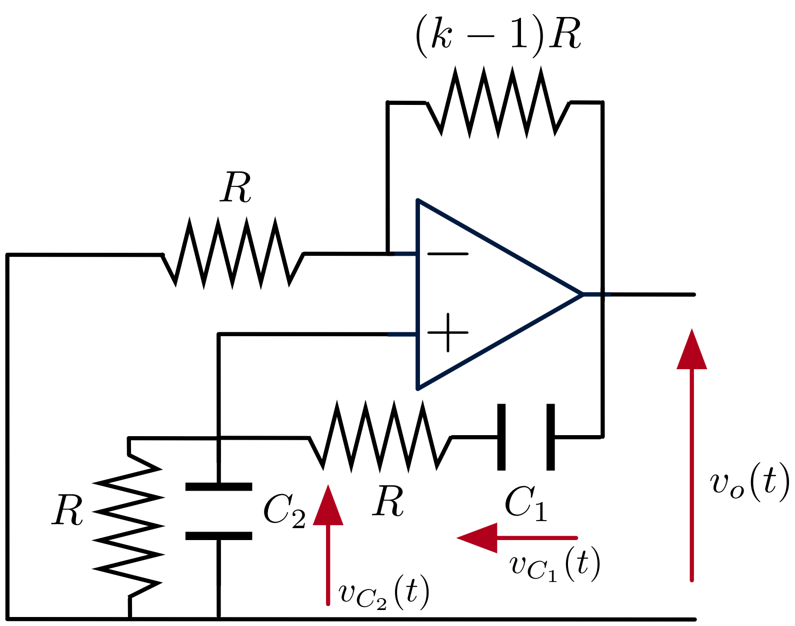

Figure 2: Wien oscillator.

(a) Let $x(t) = \begin{bmatrix} v_{C_1}(t) \\ v_{C_2}(t) \end{bmatrix}$ denote the state vector, and $y(t) = v_o(t)$ the output of the Wien oscillator circuit (notice that there is no input).

Show that its state space representation is given by

$$\dot{x}(t) = \begin{bmatrix} -\frac{1}{RC_1} & \frac{1-k}{RC_1} \\ \frac{1}{RC_2} & \frac{k-2}{RC_2} \end{bmatrix} x(t),$$
$$y(t) = \begin{bmatrix} 0 & k \end{bmatrix} x(t).$$

(b) If $C_1 = C_2 = C$, determine the range of values for the parameter k for which the resulting system is stable, asymptotically stable, or unstable.

6. Consider an LTI system whose state $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ evolves according to

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

where $u(t)$ is an external input.

(a) Does there exist a control input u (as a function of time) such that we can drive the system state from $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $x(2\pi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$?

(b) Consider the following piecewise constant input:

$$u(t) = \begin{cases} u_1 & \text{if } 0 \le t \le \frac{2\pi}{3}; \\ u_2 & \text{if } \frac{2\pi}{3} \le t \le \frac{4\pi}{3}; \\ u_3 & \text{if } \frac{4\pi}{3} \le t \le 2\pi. \end{cases}$$

Do there exist u_1, u_2 and u_3 such that we can drive the system state from $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $x(2\pi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$?

Hint: Note that the system's "A" matrix is in the form of the one in Question 3(a).

7. (a) Consider a system whose state evolves according to

$$\dot{x}_1(t) = x_2(t),$$
$$\dot{x}_2(t) = u(t),$$

where $u(t)$ is a control input. Let $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. Do there exist u_1 and u_2 such that $u(t) = u_1 t + u_2$ can drive the system state from $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $x(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

(b) Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. If (A, B) is controllable, would (A^2, B) be controllable as well? Justify your answer.

8. Consider the following two systems:

$$\text{system } S\text{:} \quad \dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$, and

$$\text{system } \hat{S}\text{:} \quad \dot{\hat{x}}(t) = A^\top \hat{x}(t) + C^\top \hat{u}(t),$$
$$\hat{y}(t) = B^\top \hat{x}(t) + D^\top \hat{u}(t),$$

where $\hat{x}(t) \in \mathbb{R}^n$, $\hat{u}(t) \in \mathbb{R}^p$ and $\hat{y}(t) \in \mathbb{R}^m$.

(a) Show that S is controllable if and only if $\hat{S}$ is observable.

(b) Show that S is observable if and only if $\hat{S}$ is controllable.

9. Consider the transfer function

$$G(s) = \frac{1}{(s+1)(s+2)}.$$

(a) Is (A, B, C, D) with

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0,$$

a realization of $G(s)$? Is the system with the matrices (A, B, C, D) above controllable and observable?

(b) Is (A, B, C, D) with

$$A = \begin{bmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix}, \; D = 0,$$

also a realization of $G(s)$? Is the system with the matrices (A, B, C, D) above controllable and observable?

(c) Comment on the effect that lack of controllability or observability may have on the realization of a transfer function.

10. OPTIONAL: Consider the controllability Gramian

$$W_c(t) = \int_0^t e^{A\tau} B B^\top e^{A^\top \tau} d\tau \in \mathbb{R}^{n \times n}.$$

Show that if it is invertible for a particular $\bar{t}$, i.e., $W_c(\bar{t}) \succ 0$, then it is invertible for any t, i.e., $W_c(t) \succ 0$, for all $t \in \mathbb{R}$.

Note: The fact that $W_c(t) \succ 0$, for all $t \geq \bar{t}$ is easier to show compared to the case where $t < \bar{t}$.