C21 Nonlinear Systems

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Scope and objectives

These lecture notes are concerned with the problem of analysis andt control design for nonlinear dynamical systems. Unlike linear systems, whose analysis is based at a large extent to linear algebra and complex analysis tools, nonlinear systems require a wide range of tools to analyze and regulate their evolution. In this direction, two main topics are covered:

- **Lyapunov stability analysis:** This constitutes an intuitive approach to analyze stability and convergence of dynamical systems without explicitly computing the solutions of the differential equations that govern their behaviour. This method forms the basis of much of modern nonlinear control theory and also provides a theoretical justification for using locally linear control techniques.
- **Passivity and linearity:** These constitute two important classes of dynamical system for which the application of Lyapunov stability theory can be simplified. We capitalize on this to analyze interconnected systems in terms of their stability properties. In particular, we derive the so called circle criterion, which provides an extension of the Nyquist stability criterion to nonlinear systems consisting of the feedback interconnection of a linear plant and a nonlinear controller.

Analysis and control of nonlinear dynamical systems is an important area of research that covers a wide range of topics that go beyond the ones outlined above. Some of these include input-output stability theory (which considers the system as a "black box" operator that transforms inputs to outputs, and investigates the stability properties of this operator using functional analysis) and differential geometry (which extends observability and controllability concepts to nonlinear systems, and provides the basis for feedback linearization, a technique that achieves linearization using a nonlinear transformation and nonlinear state feedback). These topics are outside the scope of these notes; further details can be found in the reading material below.

Learning outcomes

- Understand the notions of equilibrium and invariance of nonlinear systems;
- Understand the different notions of equilibrium stability;
- Analyze stability of (autonomous) nonlinear systems using the indirect (linearization) and the direct Lyapunov's methods;
- Analyze invariance of (autonomous) nonlinear systems using La Salle's invariance principle;
- Stability and convergence of time-varying nonlinear systems (generalization of La Salle's invariance principle based on Barbalat's lemma);
- Construct Lyapunov functions for linear and passive systems;
- Circle criterion for stability of linear systems with nonlinear feedback.

Lecture notes

These lecture notes will be availbale on Canvas. Any comments or corrections shall be sent to `kostas.margellos@eng.ox.ac.uk`

Recommended text

- K Astrom & R Murray *Feedback Systems: An Introduction for Scientists and Engineers* Princeton U.P., 2008. Chapter 4.
- S Sastry *Nonlinear Systems: Analysis, Stability, and Control* Springer-Verlag New York, 1999. Chapters 4 & 5.
- H Khalil *Nonlinear Systems* 2nd edition, Prentice-Hall, 1996. Chapters 4 & 6.

Other reading

- J-J Slotine and W Li *Applied Nonlinear Control* Prentice-Hall, 1991.
- M Vidyasagar *Nonlinear Systems Analysis* 2nd edition, Prentice-Hall, 1993.

Acknowledgements: The course follows the indicated chapters from the recommended texts which, however, provide a more advanced treatment of the theory. Special thanks to Professor Mark Cannon for sharing his lecture notes on the topic. The current version of the notes has been influenced by his material. Moreover, the first two chapters have benefited from the handouts of the course "Signal and System Theory II", taught by Prof. John Lygeros at ETH Zurich. Special thanks to Luke Rickard for proof-reading an earlier version of these notes and for providing several constructive comments.

Contents

1 Introduction to nonlinear dynamical systems

There is a plethora of theoretical and numerical tools to analyze and control linear dynamical systems. However, it is still of paramount importance to consider the problem of analysis and control when it comes into nonlinear systems as well. This is mainly for two basic reasons:

1. Most physical systems are nonlinear. Apart from inherent nonlinearities due to the physical laws that govern their behaviour (e.g., in circuit theory elements like diodes, transistors, etc, with nonlinear characteristics), nonlinearities appear often due to actuation limits or other saturations imposed to linear systems.

2. Even though linearization of a nonlinear system is a methodology that can be employed to approximate a nonlinear system by a linear one it might be insufficient for control purposes. This is since (by construction) linearization is an accurate approximation only in the vicinity of a nominal trajectory or operating point of the nonlinear system under study.

In this chapter we first introduce the general form of nonlinear dynamical systems considered in these notes. In particular, we employ the state-space description of a nonlinear system as a modelling formalism, and focus on the specific class of *autonomous* nonlinear systems that will occupy most of our developments. We will then introduce and provide formal definitions of three main notions pertaining nonlinear systems: invariant sets, equilibria, and limit cycles. Finally, we will contrast nonlinear systems with linear ones, to better appreciate the necessity for a different treatment. It should be noted that our analysis will be entirely in continuous time.

Basic notation: We will be using $\mathbb{R}$ and $\mathbb{C}$ to denote the set of real and complex numbers, respectively. We denote by $\mathbb{R}^n$ the set of n-dimensional vectors, and by $\mathbb{R}^{m\times n}$ the set of $m \times n$ matrices with real entries (for arbitrary m and n). We will use $t \in \mathbb{R}$ for the continuous time variable. By $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ we denote a function that takes as input a vector in $\mathbb{R}^n$ and returns a vector in $\mathbb{R}^m$.

1.1 Modeling formalism

Consider a dynamical system (plant) where at every time instance $t \in \mathbb{R}$, $u(t) \in \mathbb{R}^m$ denotes the input (actuation commands) to this system, and $y(t) \in \mathbb{R}^p$ its output (sensor measurements). We will be concerned with the analysis of dynamical systems that are nonlinear and can be represented in state-space form.

We say that a nonlinear, time-varying system is in *state space* form if it can be represented by

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t), \\ y(t) &= h(x(t), u(t), t), \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is referred to as the system state, and f and h are nonlinear functions of $x(t)$, $u(t)$ and (possibly) the time t.

We will refer to the function f as dynamics or vector field. The system state $x(t)$ can be thought of as an "internal" vector, whose elements correspond to physical quantities that change over time, hence their evolution is described by means of ordinary differential equations (ODEs). In these notes we will be mostly concerned with the evolution of the system state $x(t)$, and assume that the entire state is available for the purpose of control design. As such, the output equation will either become trivial ($y(t) = x(t)$), or not necessary for these developments. With a slight abuse of terminology, unless needed for other purposes, we will thus only write the differential equation of the state when referring to its state-space form.

If it happens that the vector field f does not depend explicitly on time, we say that the system is time-invariant. If in addition, it depends neither on t, nor on $u(t)$, we say that the system is autonomous, and simply represent it by $\dot{x}(t) = f(x(t))$. All results in the remainder of this chapter will refer to autonomous nonlinear systems.

A nonlinear system is autonomous if it can be represented by

$$\dot{x}(t) = f(x(t)),$$

i.e., if it does not depend explicitly on time, and is not affected by an external input vector.

When it comes to ODEs capturing the behaviour of systems it is natural to question whether solutions to these equations are well-defined, i.e., whether they exist and are unique. For linear systems this is always the case, and we can write the associated solutions in closed form (for autonomous systems this would be the zero-input transition that involves the state transition matrix/matrix exponential). For nonlinear systems this is not always the case. To this end, we will assume throughout that the vector field f is a Lipschitz continuous function on $\mathbb{R}^n$. See Appendix 5.1 for the definition of Lipschitz continuity, as well as further details on continuity in general. Under this assumption, for any initial condition (t_0, x_0) (time-state pair), there exists a unique continuous solution $x(\cdot) : \mathbb{R} \to \mathbb{R}^n$ that is compatible with the initial condition and satisfies the dynamics of the system, i.e.,

$$\begin{aligned} x(t_0) &= x_0, \\ \dot{x}(t) &= f(x(t)), \text{ for all } t \in \mathbb{R}. \end{aligned}$$

Note that when we will refer to the state solution $x(t)$ of an autonomous nonlinear system, we will imply the solution of the associated ODE starting at a given initial state $x(t_0) = x_0$.

Consider the following example of a nonlinear system; we will revisit this example when introducing additional concepts in the sequel.

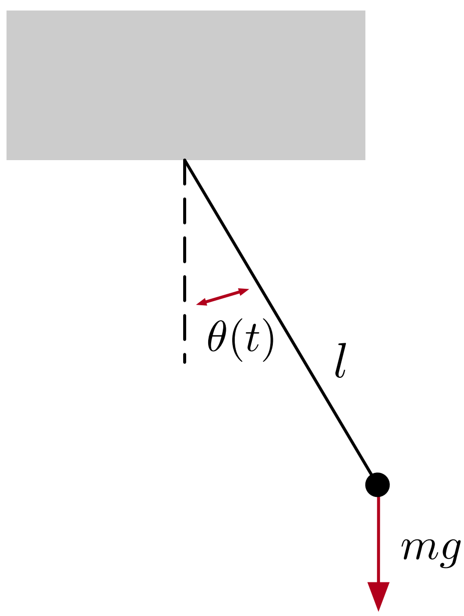

Figure 1: Pendulum with mass m.

Example 1. *Consider the pendulum illustrated in Figure 1, i.e., a mass m hanging from a weightless spring with length l. Initially the pendulum creates an angle with the vertical axis and has some angular velocity. If it is released from its initial position this angle will change as a function of time, and we denote it by $\theta(t)$. Provide a state-space description of the pendulum's motion.*

Solution: *We start by describing the motion of the pendulum, that is the evolution of $\theta(t)$. To this end, note that the pendulum performs a rotational motion with $\dot{\theta}(t)$ being its angular, and $l\dot{\theta}(t)$ its linear velocity. The pendulum mass experiences its weight mg, as well as a friction force $dl\dot{\theta}(t)$ (proportional to the linear velocity) with direction opposing its motion, with d being the friction dissipation constant. By Newton's law of motion we have that*

$$ml\ddot{\theta}(t) = -dl\dot{\theta}(t) - mg\sin\theta(t),$$

where $\ddot{\theta}(t)$ is the angular acceleration, and $mg\sin(\theta(t))$ denotes the gravity force component along the direction of motion. This is a second order ordinary differential equation (ODE) with respect to $\theta(t)$.
For every time instance t the motion of the pendulum is captured by its angular position and velocity; we can thus set as state

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix} \in \mathbb{R}^2,$$

and notice that $\dot{x}_1(t) = \dot{\theta}(t) = x_2(t)$ and $\dot{x}_2(t) = -\frac{d}{m}x_2(t) - \frac{g}{l}\sin x_1(t)$. Under these variable assignments, we are able to represent the second order ODE more compactly as

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{d}{m}x_2(t) - \frac{g}{l}\sin x_1(t) \end{bmatrix}.$$

Notice that this is a nonlinear (due to the trigonometric function), autonomous (there is no input, and time t does not appear as an explicit argument) system.

1.2 Invariant sets, equilibria, and limit cycles

Consider an autonomous nonlinear system. There are subsets of the system's state space that exhibit specific properties that become important when it comes to analyzing their long-term evolution (we will see in the next chapters their relevance to the notion of stability). In the sequel we analyze three important notions that characterize subsets of nonlinear systems' states, namely, invariant sets, equilibria and limit cycles.

Invariant sets

Denote by $x(t_0) = x_0$ the initial state of an autonomous nonlinear system, whose evolution starts at time t_0.

> A set of states $S \subseteq \mathbb{R}^n$ is said to be an invariant set if for any initial state $x_0 \in S$,
>
> $$x(t) \in S, \text{ for all } t \geq t_0,$$
>
> where $x(t)$ denotes the solution of $\dot{x}(t) = f(x(t))$ starting at x_0.

In words, we say that a set of states S is invariant, if starting inside this set ($x_0 \in S$) implies that the solutions of the ODE stays within it for all positive time instances. We can thus think of invariant sets as "non-escape" regions of the state-space.

Equilibria

An important class of invariant sets is the so called set of equilibrium points or simply equilibria. These are points so that that if we start from them, we stay there for all future time instances.

> A state $x^\star \in \mathbb{R}^n$ is called an equilibrium point of $\dot{x}(t) = f(x(t))$ if
>
> $$x_0 = x^\star \implies x(t) = x^\star, \text{ for all } t \geq t_0.$$

It follows that for autonomous systems this is equivalent to the nonlinear algebraic equation $f(x^\star) = 0$, i.e., the vector field vanishes when evaluated at an equilibrium $x^\star$. A direct implication of the equilibrium definition is that if $x^\star$ is an equilibrium point, then the set comprising of this point $\{x^\star\}$ is invariant. This follows from the invariant set definition by taking $S = \{x^\star\}$ (singleton set).

Example 2. *Consider the state-space description derived for the pendulum's motion in Example 1. Compute all equilibrium points of this system.*

Solution: *Consider the dynamics of the pendulum in state-space form as obtained in Example 1. By definition, it follows that a point* $x^\star = \begin{bmatrix} x_1^\star \\ x_2^\star \end{bmatrix}$ *is an equilibrium of the system if the dynamics vanish when evaluated at that point. Therefore,*

$$\begin{bmatrix} x_2^\star \\ -\frac{d}{m}x_2^\star - \frac{g}{l}\sin x_1^\star \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x_2^\star = 0 \text{ and } \sin x_1^\star = 0$$
$$\implies x^\star = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } x^\star = \begin{bmatrix} \pi \\ 0 \end{bmatrix}.$$

To be mathematically precise, $\sin x_1^\star = 0$ *implies that* $x_1^\star = k\pi$, $k = 0, 1, 2, \ldots$. *As a result we would have an infinite number of equilibria. However, for the pendulum only the two equilibria mentioned above are physically realizable.*

Example 2 illustrates that a system may have multiple, potentially infinite, equilibria. For nonlinear systems in particular, equilibria might be isolated (as in the pendulum example above). This is in contrast to linear systems where equilibria form a linear subspace (sometimes the only equilibrium might be the origin). Note that "isolated" is to be understood in the following sense: an equilibrium point is isolated if there is some neighbourhood around it, within which no other equilibrium point lies. We discuss this difference from linear systems further in the next section.

When it comes into analyzing the system behaviour around an equilibrium point $x^\star$, it is often algebraically convenient to "shift" it first to the origin. This involves a coordinate change, to define the new state vector

$$z(t) = x(t) - x^\star, \text{ for all } t \in \mathbb{R},$$

which acts as an error state, capturing the deviation of the state $x(t)$ from $x^\star$. The dynamics of the error system with state $z(t)$ have an equilibrium at the origin. To see this, define $\tilde{f}(z(t)) = f(z(t) + x^\star)$, and notice that

$$\dot{z}(t) = \dot{x}(t) = f(x(t)) = f(z(t) + x^\star) = \tilde{f}(z(t)),$$

where the first equality is since $\dot{x}^\star = 0$ ($x^\star$ is constant), and the last one is due to the definition of $\tilde{f}$. If a point $z^\star$ is an equilibrium for $\tilde{f}$ we have

$$\tilde{f}(z^\star) = f(z^\star + x^\star) = 0 \implies z^\star + x^\star = x^\star \implies z^\star = 0,$$

where the implication is since f vanishes when its argument becomes $x^\star$ (as this is its equilibrium). Therefore, $z^\star = 0$ is the equilibrium of the error dynamics, and hence analyzing the properties of the origin as an equilibrium point if this is more convenient, would be without loss of generality.

Limit cycles

Limit cycles are related to the notion of periodic orbits.

> A state solution $x(\cdot)$ of an autonomous nonlinear system is called periodic orbit if there exists $T > 0$ such that
>
> $$x(t + T) = x(t), \text{ for all } t \geq t_0.$$

A periodic orbit, as the name suggests, characterizes a periodic behaviour of a state solution, i.e., of a trajectory of the underlying system. Note that state solutions depend on the initial state, hence for different initial states different orbits may be obtained. Periodic orbits are obtained for systems of dimension $n \geq 2$. Equilibria give rise to trivial periodic orbits. To see this, notice that starting at an equilibrium point the system solution stays at it for all future time instances. As such the resulting solution is constant, implying that it satisfies the periodic orbit definition for any $T > 0$.

Limit cycles are nontrivial periodic orbits that are also isolated. This means that all limit cycles are periodic orbits but not vice versa. Similarly to the use of the term "isolated" for equilibria, a periodic orbit is isolated if there exists a neighbourhood around it (i.e., around the state solution/trajectory that constitutes a periodic orbit) within which no other periodic orbit exists. This is in contrast to linear systems; we illustrate this by means of an example in the next section.

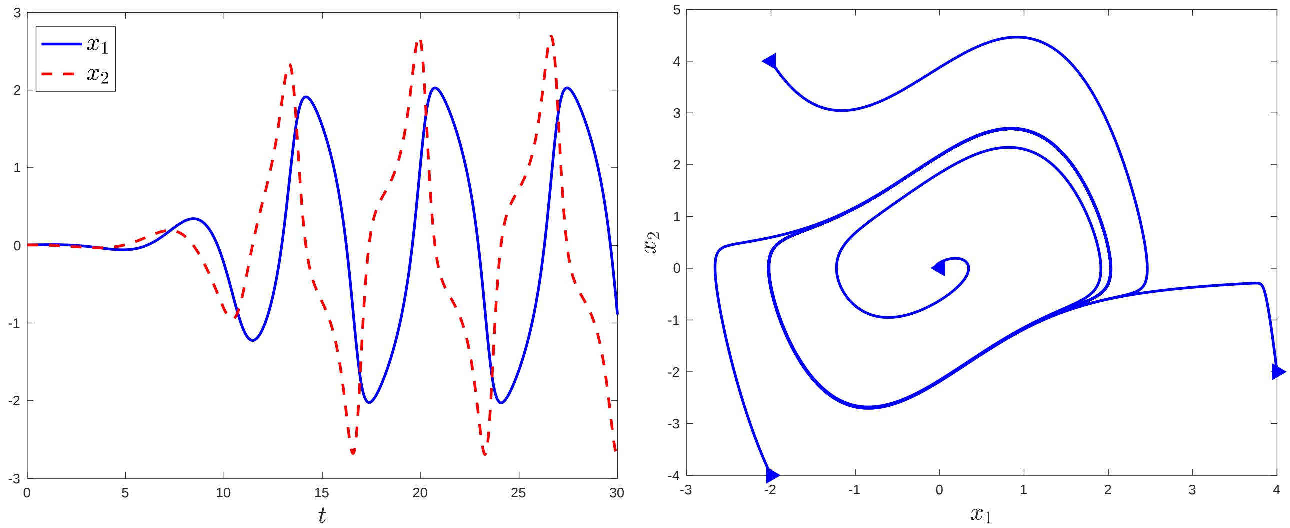

Figure 2: Van der Pol's oscillator with $\mu = 1$. Left panel: State solutions as functions of time from initial condition $x_1(0) = x_2(0) = 0.05$. Right panel: Phase portrait (x_1 vs x_2) for four different initial states, corresponding to the "triangles". The direction of these triangles shows the evolution of the system if initialized at these points. It can be observed that all these trajectories tend towards the limit cycle.

Example 3. *Consider the so called Van der Pol's oscillator (conceived originally to study electric circuits employing vacuum tubes), which characterizes an oscillating system with nonlinear damping encoded by the following differential equation.*

$$\ddot{\theta}(t) - \mu(1-\theta^2)\dot{\theta}(t) + \theta(t) = 0, \text{ where } \mu > 0.$$

Write the system in state-space form and determine its equilibria.

Solution: *Consider as state* $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$. *Notice that* $\dot{x}_1(t) = x_2(t)$, *while by means of the Van der Pol's ODE we have that* $\dot{x}_2(t) = \mu(1 - x_1(t)^2)x_2(t) - x_1(t)$. *The system's state-space representation is then given by*

$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ \mu(1 - x_1(t)^2)x_2(t) - x_1(t) \end{bmatrix}.$$

To determine the equilibria of this system let $x^\star = \begin{bmatrix} x_1^\star \\ x_2^\star \end{bmatrix}$ *be a candidate equilibrium, evaluate the system dynamics at this point and equate them with* *zero. This leads to*

$$\begin{bmatrix} x_2^\star \\ \mu(1-(x_1^\star)^2)x_2^\star - x_1^\star \end{bmatrix} = 0 \implies x_1^\star = 0 \text{ and } x_2^\star = 0.$$

Therefore, the system has a unique equilibrium point, $x^\star = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Simulating the Van der Pol oscillator, it can be observed that even if we start in the vicinity of the origin (equilibrium point) the system trajectories move away from it (see Figure 2). This behaviour is related to the equilibrium's stability properties; we will formally define the notion of stability in the next chapter. Moreover, inspecting the right panel of Figure 2 it can be observed that irrespective of the initial state, all state trajectories tend towards a single closed curve, which is known to be a limit cycle for this problem.

1.3 Differences with linear systems

Nonlinear dynamical systems exhibit several differences with linear ones. Here, we will summarize the main ones that we have already encountered. The comparison will refer to autonomous nonlinear systems $\dot{x}(t) = f(x(t))$, that will in turn be contrasted to autonomous linear systems, namely, systems of the form

$$\dot{x}(t) = Ax(t),$$

where $A \in \mathbb{R}^{n\times n}$ is the associated state-space matrix. The following main differences have been observed:

1. **Existence of solutions.** For autonomous linear systems a unique, continuous solution to the dynamics always exists. Moreover, we can compute this solution in closed form, through the state transition matrix (matrix exponential). In nonlinear systems, this is not the case. To ensure existence and uniqueness of solutions we need the vector field $f(\cdot)$ to be a Lipschitz continuous function. Otherwise, there are examples where multiple or no solutions exist. Moreover, even if a solution exists and is unique, in general it cannot be computed analytically and we rely on numerical simulation methods.

2. **Isolated equilibria.** If $x^\star$ is an equilibrium point of an autonomous linear system, then we must have that $Ax^\star = 0$. In other words, the set of all equilibria is given by the null space of matrix A; recall that $\mathrm{Null}(A) = \{x \in \mathbb{R}^n : \ Ax = 0\}$. Either this set is trivial (singleton set if A is non-singular) and in this case the origin is the only equilibrium point, or it is non-trivial and forms a linear subspace.

   In the non-trivial case, for any (arbiltrarily small) neighbourhood around an equilibrium point there necessarily exists some other equilibrium, as part of the equilibrium subspace $\{x \in \mathbb{R}^n : \ Ax = 0\}$ will pass through that neighbourhood. This is in contrast to nonlinear systems, as we have seen by means of Example 2 that nonlinear systems could have non-trivial, isolated equilibria.

3. **Limit cycles.** Linear systems may exhibit either trivial periodic orbits (e.g., solutions constant at equilibrium points), or non-trivial periodic orbits that are, however, not isolated. As a result they do not constitute limit cycles.

   To gain a better understanding on why non-trivial periodic orbits in linear systems are not isolated, consider as an example the following linear system.

$$\dot{x}(t) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x(t), \ \omega \neq 0 \Longrightarrow x(t) = \begin{bmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{bmatrix} x_0,$$

   where x_0 denotes the initial state. The state-space matrix is diagonalizable with distinct eigenvalues that are imaginary (and form a conjugate pair). We can then directly compute the state transition matrix associated with this matrix (through diagonalization), and verify that its state solution is given by the aforementioned expression. Figure 3 shows a phase portrait (x_1 vs x_2) for different initial states corresponding to the "triangles", using $\omega = 1$. Notice that for each initial state we get a different periodic orbit (circular in this case). The magnitude of the initial state affects the radius of the orbit. As such, in any (arbitrarily small) neighbourhood around any orbit there will always be some other one corresponding to another initial condition. Pictorially, for any shaded area around an orbit, other orbits (e.g., the ones with dotted lines) can always be found inside. This supports the claim that periodic orbits in linear systems are not isolated.

   In nonlinear systems periodic orbits may be isolated and hence form limit cycles. The magnitude of the limit cycle might be independent of the initial state. As an example see the Van der Pol oscillator of Figure 3 (right panel).

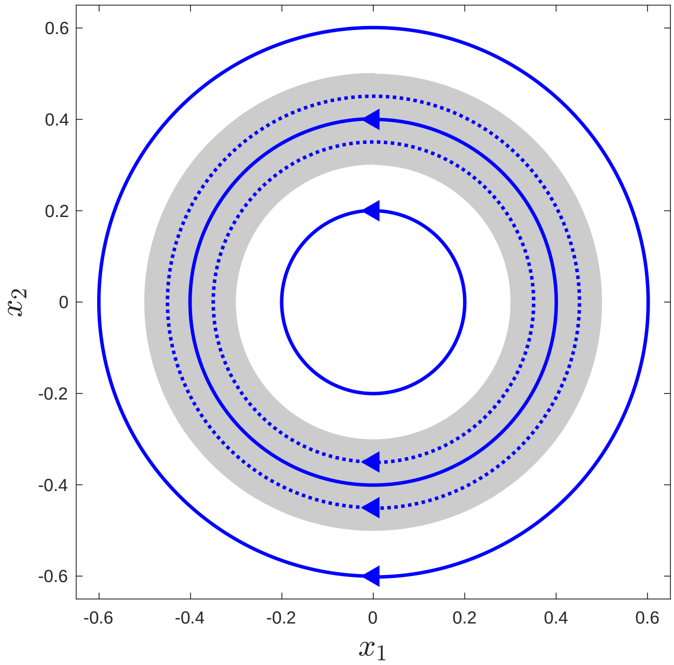

Figure 3: Periodic orbits of a marginally stable linear system (blue lines) corresponding to different initial states (triangles). The magnitude of the initial state vector affects the radius of the orbit. In any neighbourhood around an orbit (grey shaded area), another one can be found (orbits with dotted lines).

Besides these points, linear and nonlinear dynamical systems exhibit several other differences. Notably, when an autonomous linear system is stable (in a sense that would be made precise in the sequel) then it is stable for any initial condition. This property is not true for nonlinear systems, and as a result requires a rapprochement of stability and development of different methods to assess it. This will be the focus of the next chapter.

In case the underlying system is non-autonomous, and is affected by some input vector, linear systems still exhibit appealing properties. If we have a non-autonomous linear system of the form $\dot{x}(t) = Ax(t) + Bu(t)$, where $B \in \mathbb{R}^{n \times m}$, then:
(i) If we apply a sinusodial input at a given frequency, then the state solution will also contain sinusoids at the same frequency.
(ii) The principle of superposition holds, i.e., if $u(t) = a_1 u_1(t) + a_2 u_2(t)$, where a_1, a_2 are real scalar coefficients and $u_1(t), u_2(t)$ two other input vectors, then $x(t) = a_1 x_1(t) + a_2 x_2(t)$, where $x_1(t)$ and $x_2(t)$ are the state solutions corresponding to $u_1(t)$ and $u_2(t)$, respectively.
(iii) If all eigenvalues of the state-space matrix A have negative real part (i.e., autonomous counterpart of the system is asymptotically stable) and the input we

apply remains bounded, then the state solution remains bounded as well.
However *none* of these properties is in general true for nonlinear systems. These issues, however, will not be further discussed in these notes.

1.4 Summary

This chapter introduced some main notions of nonlinear dynamical systems, and contrasted them with linear ones. Its main learning outcomes can be summarized as follows:

1. Autonomous, nonlinear systems: Systems that are time-invariant and are not affected by inputs. In state-space form they can be written as

$$\dot{x}(t) = f(x(t)).$$

State solutions to these systems from an initial state $x(t_0) = x_0$ are well-defined (exist and are unique) if the vector field f is Lipschitz continuous.

2. Invariant sets, equilibria, and limit cycles: All these notions refer to the long-term evolution of the system. They are defined as follows.

Invariant sets: A set of states $S \subseteq \mathbb{R}^n$ is said to be an invariant set if for any initial state $x_0 \in S$,

$$x(t) \in S, \text{ for all } t \geq t_0,$$

Equilibria: A point $x^\star$ is called an equilibrium of $\dot{x}(t) = f(x(t))$, if

$$x_0 = x^\star \implies x(t) = x^\star, \text{ for all } t \geq t_0,$$

which is equivalent to $f(x^\star) = 0$.

Limit cycles: A limit cycle is a periodic orbit that is also isolated. Periodic orbits have the property that there exists $T > 0$ such that

$$x(t+T) = x(t), \text{ for all } t \geq t_0.$$

3. Differences with linear systems: Three main differences between autonomous nonlinear systems and autonomous linear systems of the form $\dot{x}(t) = Ax$ have been observed:

   (i) For linear systems a unique, continuous solution always exists. Moreover, it can be characterized in closed form. For nonlinear systems we need to impose conditions on the vector field f (Lipschitz continuity) for existence and uniqueness of solutions, while closed form solutions cannot be computed in general.

   (ii) Unlike linear systems, nonlinear ones may have nontrivial equilibria that are also isolated.

   (iii) Unlike linear systems, nonlinear systems may have periodic orbits that are also isolated, i.e., limit cycles.

2 Stability analysis

Consider an autonomous nonlinear system of the form

$$\dot{x}(t) = f(x(t)),$$

where $x(t) \in \mathbb{R}^n$, and assume that state trajectories from any initial state $x(t_0) = x_0$ are well defined. We have seen that invariant sets constitute non-escape regions of the state space, so if a trajectory enters such region, it stays therein forever. If we think these as "safe" portions of the state space, it is of benefit to assess whether a system's solution converges to an invariant set or at least stay close to it.

In this chapter we will provide a systematic framework to address this problem for a specific subclass of invariant sets, namely, equilibria (recall that equilibria are invariant sets themselves). To achieve this, we address the following questions:

1. Can we formalize the concept of staying close or converging to an equilibrium point? This is directly related to the notion of stability. We will introduce different types of stability for equilibria of autonomous, nonlinear systems. In contrast to linear systems, if a nonlinear system is stable, it is not necessarily stable from all initial states. To accommodate this issue we will distinguish between local and global stability definitions, and introduce the concept of the domain of attraction of a given equilibrium point.
2. Given a nonlinear system, can we assess whether its equilibria are stable (in some sense to be made precise in the sequel)? Assessing the stability properties of autonomous linear systems simply requires inspecting the eigenvalues of the state space matrix. For nonlinear systems this is a much more challenging task. We will provide systematic tools to perform this using Lyapunov's indirect and direct stability methods (also known as first and second method of Lyapunov).

2.1 Types of stability

In the following, every time we will refer to an equilibrium point $x^\star$, we will imply an equilibrium of $\dot{x}(t) = f(x(t))$. Without loss of generality, as we focus on time-invariant systems, we consider that $t_0 = 0$, and hence $x_0 = x(0)$.

Stability

Informally stability of an equilibrium point implies that if we start close to it, then we stay close to it for all subsequent time instances. To quantify the notion "close" we use the Euclidean norm $\|\cdot\|$ as a distance metric.

An equilibrium $x^\star$ is called stable if for any $\epsilon > 0$, there exists $\delta > 0$, such that

$$\|x_0 - x^\star\| < \delta \implies \|x(t) - x^\star\| < \epsilon, \text{ for all } t \geq 0.$$

An equilibrium is called unstable if it is *not* stable.

Figure 4 provides a pictorial illustration of the equilibrium stability definition. With reference to this figure, let $\epsilon > 0$ be the radius of a neighbourhood around an equilibrium point $x^\star$ (we refer to it as ϵ-ball), within its interior we wish our system to stay. The stability definition implies that our system is cast to be stable if there exists a possibly smaller ball with radius $\delta > 0$, such that if we start from this inner ball, i.e., $\|x_0 - x^\star\| < \delta$ then the state trajectory is necessarily contained in the outer one, i.e., $\|x(t) - x^\star\| < \epsilon$ for all $t \geq 0$, which implies that it stays within the predefined neighbourhood around the equilibrium. Note that the state does not have to return to the equilibrium but rather to stay close to it. In other words, if an equilibrium is stable, starting δ-close to it implies that we stay ϵ-close.

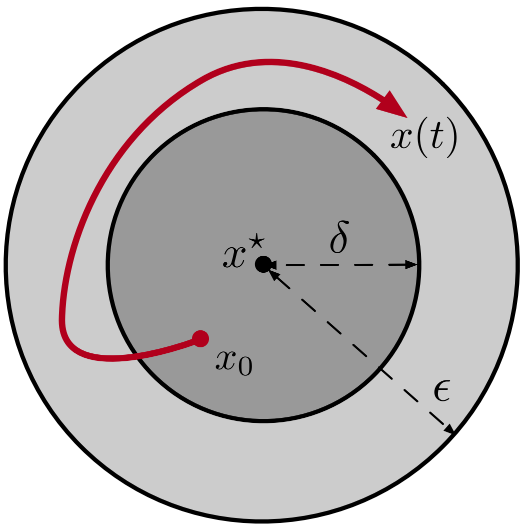

Figure 4: Pictorial illustration of the equilibrium stability definition.

Note that an equilibrium point may be unstable even if trajectories emanating from states in a neighbourhood around it do not diverge to infinity. As an example consider the Van der Pol oscillator of Example 3, where the origin is the unique equilibrium point, but all trajectories that start close to it are driven towards the limit cycle (see Figure 2). More formally, if we select the ϵ-radius ball in the stability definition to be contained in the limit cycle (in fact it suffices that this ball leaves out only part of the limit cycle), then it is not possible to find $\delta > 0$ such that if we start δ-close to the equilibrium we stay within the chosen ϵ-ball. This is since no matter how small δ is chosen, the state trajectories will converge to the limit cycle that is outside our selected ϵ-ball. As such, the equilibrium in the Van der Pol oscillator example is unstable.

Asymptotic stability

Asymptotic stability is a stronger notion compared to stability. Not only do we stay close to an equilibrium point if we start sufficiently close to it (as stability suggests), but rather converge to it asymptotically as time tends to infinity. We formalize this in the following definition.

An equilibrium $x^\star$ is called locally asymptotically stable if

1. it is stable;
2. and there exists $\delta > 0$, such that

$$\|x_0 - x^\star\| < \delta \implies \lim_{t\to\infty} x(t) = x^\star.$$

An equilibrium is globally asymptotically stable if this holds for any $\delta > 0$.

In Appendix 5.4 we discuss why both conditions are necessary in the asymptotic stability definition. It is also useful to characterize all initial states from which trajectories can start and converge to the equilibrium $x^\star$. This is the set of initial states that satisfy $\|x_0 - x^\star\| < \delta$; notice that is set is independent of t_0, as the system is time-invariant.

The set of initial states x_0 from which state trajectories can start and satisfy $\lim_{t\to\infty} x(t) = x^\star$ is called domain of attraction of the equilibrium point $x^\star$.

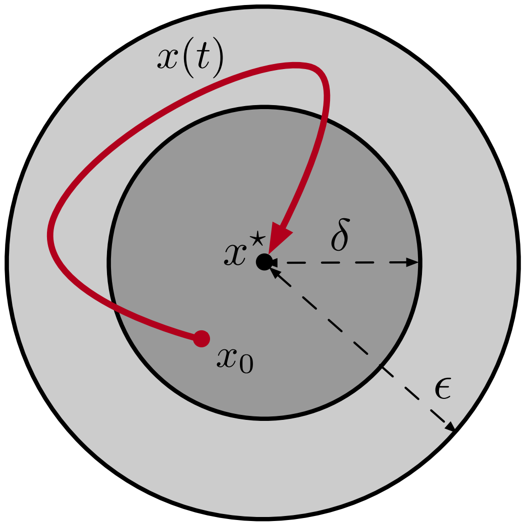

Figure 5: Pictorial illustration of the equilibrium asymptotic stability definition.

Figure 5 provides a pictorial illustration of the equilibrium asymptotic stability definition. The addition compared to Figure 4 is that now trajectories that start δ-close an equilibrium not only have to stay close to it, but converge to $x^\star$ as $t \to \infty$. The following remarks are in order.

1. By the definition of the limit, the second condition of the asymptotic stability definition is equivalent to requiring that there exists $\delta > 0$ such that for every $M > 0$, there exists some time instance $T(M, \delta)$ (we mean that this time depends on the chosen M and δ) such that

$$\|x_0 - x^\star\| < \delta \implies \|x(t) - x^\star\| < M, \text{ for all } t \geq T(M, \delta).$$

If $M \to 0$ (so that $x(t)$ tends to $x^\star$) then $T(M, \delta) \to \infty$, which witnesses the equivalence with the limit in the asymptotic stability definition.

2. Notice the distinction between local and global asymptotic stability. Local asymptotic stability essentially requires that state trajectories converge to an equilibrium point $x^\star$ from any initial state δ-close to the equilibrium, i.e., within the domain of attraction. Global asymptotic stability requires this to be the case for any $\delta > 0$, or in other words, it requires the domain of attraction to be $\mathbb{R}^n$.

3. Sometimes, when an equilibrium is globally asymptotically stable, we say that the system is globally asymptotically stable. There is no ambiguity with this slight abuse of terminology as, if an equilibrium is globally asymptotically stable, this will necessarily be the only equilibrium point of the system. To see this notice that

in the opposite case, i.e., if other equilibria are present, the domain of attraction could not be $\mathbb{R}^n$ and hence the system would not be globally asymptotically stable, as at least these other equilibria would be excluded from that domain (recall that starting at an equilibrium implies that we stay there).

Exponential stability

Exponential stability strengthens the notions of asymptotic stability by requiring that state trajectories tend to an equilibrium at a rate that is bounded by an exponential. We formalize this in the following definition.

An equilibrium $x^\star$ is called locally exponentially stable if there exist constants $\delta, M, \alpha > 0$ such that

$$\|x_0 - x^\star\| < \delta \implies \|x(t) - x^\star\| \leq Me^{-\alpha t}\|x_0 - x^\star\|, \text{ for all } t \geq 0.$$

An equilibrium is globally exponentially stable if this holds for any $\delta > 0$.

The largest constant α for which the condition of the exponential stability definition is satisfied, is called the *rate of convergence*.

For linear systems if an equilibrium is asymptotically stable, then it is also exponentially stable. This is since asymptotic stability for linear systems is equivalent to all eigenvalues of the state space matrix having negative real part. The state solution will then include exponential terms with the real part of the eigenvalues in the exponent; the smallest in magnitude real part dictates the rate of convergence.

2.2 Assessing stability

We now capitalize on the equilibria stability classification performed in the previous section and develop methodologies to assess whether a given equilibrium exhibits these properties. Our results provide tools to analyze equilibria in terms of stability and (local or global) asymptotic stability. We will not discuss exponential stability further, and only comment on extensions towards this direction.

2.2.1 Lyapunov's indirect method

Lyapunov's indirect method for stability provides a straightforward approach to assess the stability properties of an equilibrium point locally. It employs the procedure of linearization: locally, potentially in a small neighbourhood around an equilibrium point, a nonlinear system can be approximated by a linear one. Lyapunov's indirect method is then based on:

1. Linearizing the nonlinear system around an equilibrium point;
2. Analyzing the resulting linear approximation in terms of stability;
3. Using the stability conclusions of this analysis to infer information about the (local) stability properties of the original nonlinear system.

Note that any stability statement through this procedure is necessarily local, as it is based on the validity of the linearization as an approximation of the nonlinear system, which is in general accurate only in a neighbourhood around the equilibrium.

Let $x^\star$ be an equilibrium point of $\dot{x}(t) = f(x(t))$. The linearization of this system around $x^\star$ is based on a Taylor's theorem for expanding f about $x^\star$. Assuming that $f(x)$ is continuously differentiable[§], we can get a first order approximation by neglecting higher order terms in this expansion, obtaining

$$f(x(t)) \approx f(x^\star) + \frac{\partial f}{\partial x}(x^\star)(x(t) - x^\star) = \frac{\partial f}{\partial x}(x^\star)(x(t) - x^\star),$$

where the equality is since $f(x^\star) = 0$ as $x^\star$ is an equilibrium, and $\dfrac{\partial f}{\partial x}(x^\star) \in \mathbb{R}^{n\times n}$ is the Jacobian of f evaluated at $x^\star$. Let

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \; f(x(t)) = \begin{bmatrix} f_1(x_1(t), \dots, x_n(t)) \\ \vdots \\ f_n(x_1(t), \dots, x_n(t)) \end{bmatrix},$$

[§]A function is continuously differentiable if it is differentiable and its (partial) derivatives are themselves continuous. This condition ensures that the first order approximation in the Taylor's expansion by neglecting higher order terms, is accurate enough in the vicinity of $x^\star$. Specifically, the so called Peano remainder $r(x(t))$, $t \geq 0$, when higher order terms are truncated, is well defined and as $x(t)$ tends to $x^\star$, it tends to zero faster than a linear function of $x(t) - x^\star$, i.e., $\lim_{x \to x^\star} \frac{\|r(x)\|}{\|x - x^\star\|} = 0$.

$$\text{and } A = \frac{\partial f}{\partial x}(x^\star) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x^\star) & \cdots & \frac{\partial f_1}{\partial x_n}(x^\star) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x^\star) & \cdots & \frac{\partial f_n}{\partial x_n}(x^\star) \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Denoting now $z(t) = x(t) - x^\star$, we have that

$$\dot{z}(t) = \dot{x}(t) \approx A(x(t) - x^\star) = Az(t),$$

where the first equality is since $x^\star$ is constant, and the approximate equality is due to the first order approximation. Hence, the state $z(t)$ that captures the error between the nonlinear system and its linear approximation behaves like a linear system (close to $x^\star$ so that neglecting the higher order terms is a valid approximation).

Lyapunov's indirect method analyzes the stability of this error system (the linearized one), to comment on the stability properties of the original nonlinear system.

Theorem 1 (Lyapunov's indirect method). *Consider the system $\dot{x}(t) = f(x(t))$ and assume f is continuously differentiable. Consider also its linearization around an equilibrium $x^\star$ with matrix A as defined above.*
The equilibrium $x^\star$ is:

1. *Locally asymptotically stable if all eigenvalues of A have negative real part.*
2. *Unstable if A has at least one eigenvalue with positive real part.*

Theorem 1 provides a straightforward way to check stability: it only requires computing the eigenvalues of matrix A of the linearized system. However, it suffers from the fact that it is inconclusive if A has imaginary or zero eigenvalues. In such cases we cannot draw conclusions about the stability properties of the nonlinear system (it may or may not be stable) simply from its linearization. This issue does not appear in Lyapunov's direct method that will be introduced in the sequel. Example 4 illustrates this issue further.

Moreover, Theorem 1 does not offer any insight on the domain of attraction, which is influenced by the validity of the linearization.

✎ **Example 4.** *Consider the systems $\dot{x}(t) = -x^2(t)$ and $\dot{x}(t) = x^2(t)$, $t_0 = 0$, $x(0) = x_0$, and verify they admit the following state solutions, respectively,*

$$x(t) = \frac{x_0}{1 + tx_0} \text{ and } x(t) = \frac{x_0}{1 - tx_0}.$$

Use Lyapunov's indirect method to comment on the stability of these systems.
Solution: *To verify that the given expressions are indeed state solutions notice that $x(0) = x_0$ so that the initial condition is satisfied, and differentiating them with respect to time we obtain the associated ODEs.*
The point $x^\star = 0$ is the unique equilibrium for both systems. Linearization around $x^\star$ leads in both cases to $\dot{z}(t) = 0$ (as the first derivative of the dynamics is zero when evaluated at $x^\star = 0$), which has a zero eigenvalue. This in turn implies that Theorem 1 is inconclusive. However, notice that

$$\lim_{t\to\infty} \frac{x_0}{1 + tx_0} = 0 \text{ and } \lim_{t\to\frac{1}{x_0}} \frac{x_0}{1 - tx_0} = \infty.$$

This illustrates that the origin is in fact an asymptotically stable equilibrium for the first system since the state converges to it as time tends to infinity, while it is an unstable equilibrium for the second system, as state escapes to infinity in finite time, i.e., as $t \to \frac{1}{x_0}$.

✎ **Example 5.** *Consider the state space representation of the pendulum system derived in Example 1. Use Theorem 1 to comment on the stability of the system's equilibria (see Example 2), namely,*

$$x^\star = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } x^\star = \begin{bmatrix} \pi \\ 0 \end{bmatrix}.$$

Solution: *We first linearize the system around its equilibria. By the state space representation of the pendulum we obtain that*

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{d}{m}x_2 - \frac{g}{l}\sin x_1 \end{bmatrix}.$$

The linearization of the system around $x^\star = (x_1^\star \; x_2^\star)$ involves the matrix

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x^\star) & \frac{\partial f_1}{\partial x_2}(x^\star) \\ \frac{\partial f_2}{\partial x_1}(x^\star) & \frac{\partial f_2}{\partial x_2}(x^\star) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1^\star & -\frac{d}{m} \end{bmatrix}.$$

It turns out that the linearized system depends only on $x_1^\star$; for the two different values of $x_1^\star$ according to the operating points we obtain

$$x_1^\star = 0 \;\Rightarrow\; A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{d}{m} \end{bmatrix} \text{ and } x_1^\star = \pi \;\Rightarrow\; A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{d}{m} \end{bmatrix}.$$

We now calculate the eigenvalues of the A matrix that results for each equilibrium point. This can be achieved by calculating the roots of the characteristic polynomial of each matrix. We thus have

$$x_1^\star = 0 \;\Rightarrow\; \lambda^2 + \frac{d}{m}\lambda + \frac{g}{l} = 0 \;\Rightarrow\; \lambda_{1,2} = -\frac{d}{2m} \pm \frac{1}{2}\sqrt{\frac{d^2}{m^2} - \frac{4g}{l}},$$
$$x_1^\star = \pi \;\Rightarrow\; \lambda^2 + \frac{d}{m}\lambda - \frac{g}{l} = 0 \;\Rightarrow\; \lambda_{1,2} = -\frac{d}{2m} \pm \frac{1}{2}\sqrt{\frac{d^2}{m^2} + \frac{4g}{l}}.$$

We distinguish two cases according to the values of $d \geq 0$.
***Case** $d > 0$**:** If $x_1^\star = 0$, in the case that the discriminant is negative then the eigenvalues will form a complex conjugate pair with real part $-\frac{d}{2m} < 0$. In the case that the discriminant is positive and since the square root would be smaller in magnitude compared to $\frac{d}{m}$, both eigenvalues will be real and negative. As a result in any case, all eigenvalues would have negative real part. By Theorem 1 we thus have*

$$x^\star = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \text{Locally asymptotically stable.}$$

If $x_1^\star = \pi$, we have that $-\frac{d}{2m} + \frac{1}{2}\sqrt{\frac{d^2}{m^2} + \frac{4g}{l}} > 0$, hence one eigenvalue would be positive (notice it will be real). By Theorem 1 we thus have

$$x^\star = \begin{bmatrix} \pi \\ 0 \end{bmatrix} \implies \text{Unstable.}$$

Case $d = 0$***:*** *If* $x_1^\star = 0$*, then eigenvalues become imaginary, i.e.,* $\lambda_{1,2} = \pm j\sqrt{\frac{g}{l}}$*. In this case, Theorem 1 is inconclusive.*
If $x_1^\star = \pi$*, then eigenvalues become* $\lambda_{1,2} = \pm\sqrt{\frac{g}{l}}$*. Hence one of them is positive and by Theorem 1 we thus conclude that* $x^\star = \begin{bmatrix}\pi \\ 0\end{bmatrix}$ *is unstable.*

It should be noted that the Lyapunov's indirect method can be generalized to non-autonomous systems. However, in such cases the linearized system is time-varying, and deciding its stability is in general a more difficult task.

2.2.2 Lyapunov's direct method

Lyapunov's direct method aims at determining the stability properties of an equilibrium point of a nonlinear system (without inputs) without solving the differential equations describing the system. It rather capitalizes on the properties of a continuously differentiable function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ which should be such that[§]:

1. its minimum is zero and its attained at the equilibrium $x^\star$, i.e., $V(x^\star) = 0$;
2. it is positive for all other states, i.e., $V(x) > 0$ for all $x \neq x^\star$;
3. its time derivative is non-positive over the state space, i.e., $\dot{V}(x) = \frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \leq 0$ for all x. Since $\dot{x}(t) = f(x(t))$, this time derivative can be written as
$$\dot{V}(x) = \frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = \sum_{i=1}^{n} \frac{\partial V(x(t))}{\partial x_i}\dot{x}_i(t)$$
$$= \sum_{i=1}^{n} \frac{\partial V(x)}{\partial x_i} f_i(x) = \nabla V(x) f(x),$$
where $x_i(t)$, $f_i(x(t))$ denote the i-the term of $x(t)$, $f(x(t))$, respectively, and $\nabla V(x)$ (row vector) is the gradient of V. Requiring $\dot{V}(x) = \nabla V(x) f(x) \leq 0$ to be non-positive implies that V cannot increase along state trajectories (see Figure 6). The quantity $\nabla V(x) f(x)$ can be thought of as the rate of change of V along the dynamics $f(x)$, also known as the Lie derivative of V along f.

[§]Note that in the conditions $V(x)$ and its time derivative need to satisfy we will not show t as the argument of x unless necessary. This is to ease notation, but also to avoid any confusion as we would need to evaluate these conditions at any admissible state x and not only at the states visited by a system trajectory that was initialized at a specific x_0.

A function V with these properties is known as a Lyapunov function. Figure 6 provides some geometric interpretation of the Lyapunov function properties. The gradient $\nabla V(x)$ is perpendicular to the associated level set (see contour levels), pointing in the direction of increasing values of V. The condition then $\nabla V(x) f(x) \leq 0$ implies then that the angle formed by $\nabla V(x)$ and $f(x)$ is greater than or equal to $90°$. Hence, the vector field points inwards or is tangent to the level sets of V, implying that if the system finds itself within a particular level set then trajectories cannot escape that set and will evolve further inside. This directly has certain stability implications for $x^\star$.

Alternatively, a Lyapunov function can be thought of as an energy-like function with the equilibrium being at zero energy and the non-increase condition implying that from any given initial state, energy along system trajectories cannot increase.

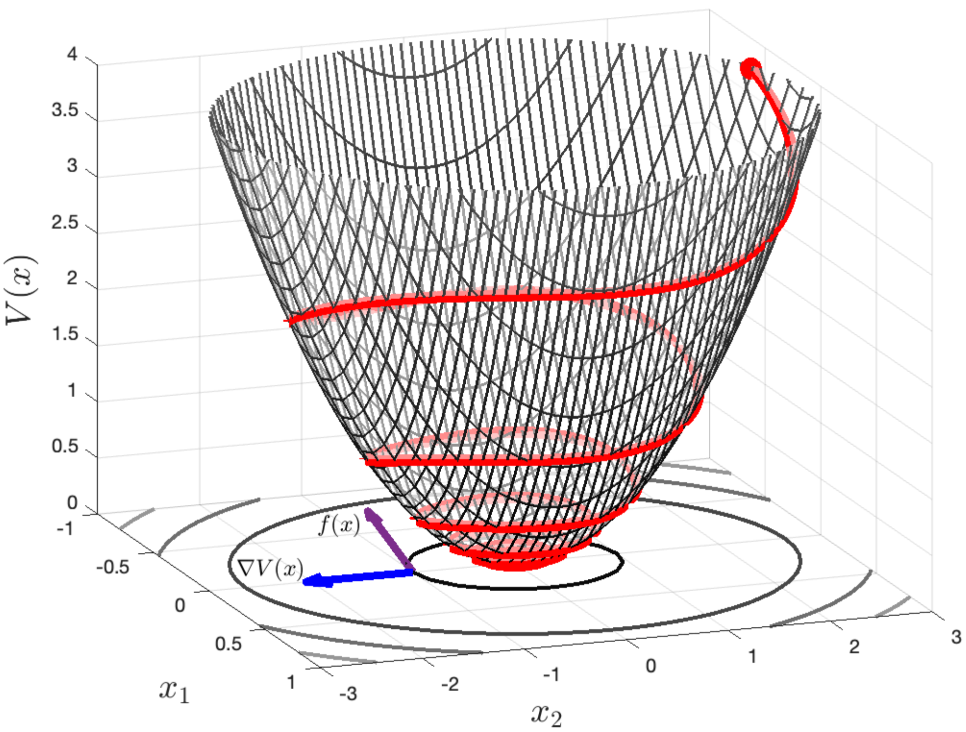

Figure 6: Pictorial illustration of a Lyapunov function. The solid red line indicates a system trajectory starting from the initial state indicated by the dot (this refers to the pendulum system of Example 1 with $d = 0.5$ and all other parameters unitary). Notice that V does not increase along this trajectory. The blue arrow indicates $\nabla V(x)$ at a given point; this is perpendicular at the associated level set and points in the direction of increasing values of V. Similarly, the purple arrow indicates the vector field $f(x)$ at the same point. Notice that the angle between these arrows is greater than $90°$; hence the vector field points inwards (or at other points may be tangent) to the level sets of V. This is aligned with the condition $\nabla V(x) f(x) \leq 0$. Vectors $\nabla V(x)$ and $f(x)$ are translated to the horizontal plane for illustration purposes.

We formalize the stability implications of a Lyapunov function below. To this end, let the equilibrium $x^\star$ belong to an open set[§].

Theorem 2 (Lyapunov's direct method – Stability). *Let $x^\star$ be an equilibrium of $\dot{x}(t) = f(x(t))$. Assume that there exists an open set $S \subseteq \mathbb{R}^n$ that contains the equilibrium, i.e., $x^\star \in S$. Assume also that there exists a continuously differentiable function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ such that*

1. *$V(x^\star) = 0$,*
2. *$V(x) > 0$, for all $x \in S$ with $x \neq x^\star$,*
3. *$\dot{V}(x) \leq 0$, for all $x \in S$.*

We then have that the equilibrium point $x^\star$ is stable.

Proof. By the definition of stability, to show that the equilibrium $x^\star$ is stable we need to show that for any $\epsilon > 0$, there exists $\delta > 0$, such that if $\|x_0 - x^\star\| < \delta$ then $\|x(t) - x^\star\| > \epsilon$, for all $t \geq 0$. To this end, fix any $\epsilon > 0$, and under the theorem's assumptions on V we will construct a $\delta > 0$ such that this statement holds.

With reference to Figure 7, let S be some open set around $x^\star$. To simplify the proof we assume that this set is "big enough" so that the boundary of the ϵ-ball around $x^\star$ is inscribed in S; otherwise, the proof would remain the same but we would need to consider an inscribed r-ball for some $0 < r \leq \epsilon$. Notice that the boundary is itself a closed set (its complement is open) and bounded (it does not extend to infinity), hence it is compact (see Appendix 5.1). By the so called Weierstrass theorem, since V is assumed to be continuous, it attains its minimum over a compact set, namely the boundary of the ϵ-ball. Denote this minimum by $\bar{c}$, i.e.,

$$\bar{c} = \min_{\{x:\ \|x - x^\star\| = \epsilon\}} V(x) > 0,$$

where $\{x :\ \|x - x^\star\| = \epsilon\}$ is the boundary of the ϵ-ball. Notice that $\bar{c} > 0$ since V is positive for all points $x \in S$ with $x \neq x^\star$ (property 2 of V). If for the sake of illustration V is a function with ellipsoidal level sets, then the boundary of the

[§]A set $S \subseteq \mathbb{R}^n$ is open if for any point $x \in S$, there exists a (possibly small) neighbourhood around that point so that the entire neighbourhood is contained in S. It is closed if its set complement is open. See Appendix 5.1 for some examples.

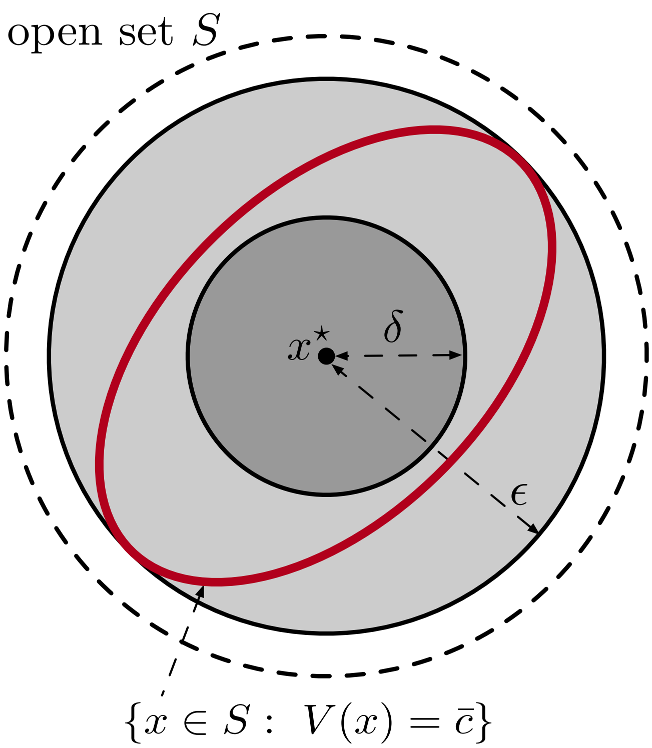

Figure 7: Pictorial illustration for the construction in the proof of Theorem 2.

$\bar{c}$-level set of V, that is $\{x \in S : \ V(x) = \bar{c}\}$, "touches" the boundary of the ϵ-ball. This is indicated by the solid red level set in Figure 7. Since V is continuous, there exists $\delta > 0$ such that for any $\|x - x^\star\| < \delta$,

$$\|V(x) - V(x^\star)\| = V(x) < \bar{c},$$

where the equality is since V is non-negative and $V(x^\star) = 0$ (properties 1 & 2 of V). We claim that this δ is the appropriate choice for the stability definition, so that any trajectory starting inside that δ-ball stays within the fixed ϵ-ball. To show this, assume for the sake of contradiction that there exists $\bar{x}$ inside that δ-ball (by the continuity statement above we would have $V(\bar{x}) < \bar{c}$), such that the trajectory starting at $\bar{x}$ reaches and then exits the boundary of the ϵ-ball at some time $T > 0$. We denote by $x(T)$ the "exit" state when the trajectory intersects the boundary of the ϵ-ball. This in turn implies that $V(x(T)) \geq \bar{c}$, as the points on the boundary of the ϵ-ball (which is taken to be open) belong to level sets of V with value greater or equal to $\bar{c}$ (equal if they also belong to the ellipsoid). By property 3 of V, we have that V is non-increasing along state trajectories, i.e., $V(x(T)) \leq V(\bar{x})$. Overall,

$$\bar{c} \leq V(x(T)) \leq V(\bar{x}) < \bar{c},$$

where the first inequality is by the value of V at the exit time, the second one is due to the non-increase of V, and the strict inequality due to continuity of V at $x^\star$. This establishes a contradiction, showing that if trajectories start from the constructed δ-ball, they will stay in the ϵ-ball. As such, $x^\star$ is stable. $\square$

In the proof of Theorem 2 we did not use the fact that V was assumed to have continuous derivatives. We keep it, however, for uniformity as this is needed in the proof of Theorem 3. We now strengthen the third property that a candidate Lyapunov function needs to satisfy to a strict inequality.

Theorem 3 (Lyapunov's direct method – Asymptotic Stability)**.** *Let $x^\star$ be an equilibrium of $\dot{x}(t) = f(x(t))$. Assume that there exists an open set $S \subseteq \mathbb{R}^n$ that contains the equilibrium, i.e., $x^\star \in S$. Assume also that there exists a continuously differentiable function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ such that*

1. *$V(x^\star) = 0$,*
2. *$V(x) > 0$, for all $x \in S$ with $x \neq x^\star$,*
3. *$\dot{V}(x) < 0$, for all $x \in S$ with $x \neq x^\star$.*

We then have that the equilibrium point $x^\star$ is locally asymptotically stable. If $S = \mathbb{R}^n$ then the equilibrium point $x^\star$ is globally asymptotically stable.

Notice that in Theorem 3 the set S plays the role of the domain of attraction. The Lyapunov function offers the means to obtain an estimate for this domain: any c-level set of V inscribed in S, namely, $\{x \in \mathbb{R}^n : \; V(x) \leq c\} \subseteq S$ would be a domain of attraction due to the non-increase condition of V. As such, any trajectory starting in that set would stay therein for all future time instances. It follows that any valid open set S should not include other equilibria besides $x^\star$ as $\dot{V}$ would then be zero at these points as well.

✎ **Example 6.** *Consider the state space representation of the pendulum system derived in Example 1. Use Lyapunov's direct method to decide on whether the equilibrium point $x^\star = (0, 0)$, for which Lyapunov's indirect method was inconclusive (see Example 5), is stable/asymptotically stable.*
Consider the open set $S = (-\pi, \pi) \times \mathbb{R}$, and as a candidate Lyapunov function

the pendulum's energy (with $mgl(1-\cos x_1)$ being the potential energy), namely,

$$V(x) = \frac{1}{2}ml^2x_2^2 + mgl(1-\cos x_1).$$

Solution: *We first check whether the assumptions of Theorem 2 on the candidate Lyapunov function are satisfied. The function V is continuously differentiable, and it also satisfies the three required conditions:*

1. *$V(x^\star) = 0$.* ✓
2. *$V(x) > 0$, for all $x \in (-\pi, \pi) \times \mathbb{R}$ with $x \neq x^\star$.* ✓
3. *Computing $\dot{V}(x)$, we obtain that*

$$\begin{aligned}\dot{V}(x) &= \frac{\partial V(x)}{\partial x_1} f_1(x) + \frac{\partial V(x)}{\partial x_2} f_2(x) \\ &= (mgl\sin x_1)x_2 + ml^2x_2\big(-\frac{d}{m}x_2 - \frac{g}{l}\sin x_1\big) \\ &= -dl^2x_2^2 \leq 0. \quad ✓\end{aligned}$$

Note that in case $d = 0$, $\dot{V}(x) = 0$ which aligns with intuition as there is no damping. Therefore, for any $d \geq 0$, $\dot{V}(x) \leq 0$, for all $x \in (-\pi, \pi) \times \mathbb{R}$.

All assumptions of Theorem 2 are satisfied, hence V is a valid Lyapunov function, and we can infer that $x^\star$ is a stable equilibrium point.
We also have to check whether Theorem 3 is also applicable. Condition 3 of Theorem 3 reuires $\dot{V}(x)$ to be negative for all points in $x \in (-\pi, \pi) \times \mathbb{R}$ with $x \neq x^\star$. However, $\dot{V}(x) = 0$ for all points with $x_2 = 0$ (not just the origin). As such, we cannot conclude asymptotic stability of $x^\star$.
Figure 8 shows an illustration of this Lyapunov function; its contour level sets are shown as projections on the $x_1 - x_2$ plane. Moreover, Figure 9 shows contour levels of the Lyapunov function for a region that encompasses more equilibria (even though not all of them are physically realizable).

Example 6 illustrates the fact that Lyapunov's direct method in Theorem 2 allows us to show that the equilibrium point under study is stable for all $d \geq 0$, even though the indirect method for the same problem was inconclusive for $d = 0$. However, Theorem 3 is not applicable, so we cannot decide whether the equilibrium

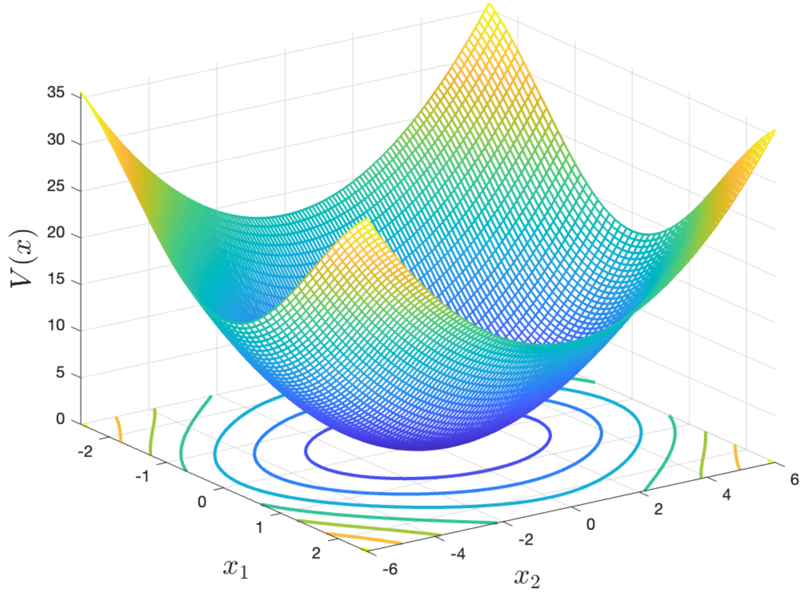

Figure 8: Lyapunov function for the pendulum system used in Example 6.

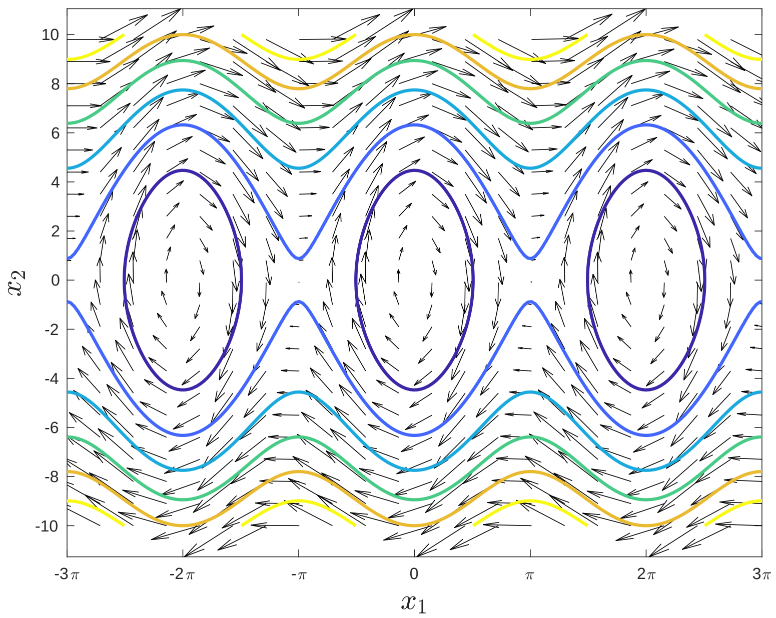

Figure 9: Contour levels of the Lyapunov function in Figure 8. The domain of x_1 is extended to $[-3\pi, 3\pi]$. This region encompasses then multiple equilibria (even though not all of them are physically realizable); see also Example 2. Notice that for $k = 0, 1, 2 \ldots$, equilibria with $x_1^\star = \pm 2k\pi$, are stable, while equilibria with $x_1^\star = \pm(2k+1)\pi$ are unstable. The arrows show the vector field for $d = 0.5 > 0$; notice that around stable equilibria these arrows form centers (orbits of decreasing magnitude) since the linearized system at these points has imaginary eigenvalues, while unstable equilibria perform like saddle points, since the linearized system at these points has real eigenvalues of opposite sign.

is asymptotically stable. It turns out that this equilibrium for $d > 0$ is in fact locally asymptotically stable as we observed in Example 5 using Lyapunov's indirect method; we will retrieve this result using the methodology developed in the next chapter, and also obtain an estimate for the domain of attraction.

Theorems 2 & 3 offer sufficient conditions for stability. Converse statements also exist in the sense that if an equilibrium is stable/asymptotically stable then a Lyapunov function exists, but with reference to Example 6, this does not have to be the same function with the one considered there. In general, Lyapunov's direct methods rely on determining a candidate Lyapunov function. This is a case dependent, and in general difficult task. For some problems, like the pendulum example, the energy of the system (due to the link between the Lyapunov function's properties and the system's energy) could be chosen as a candidate function. Note also that Lyapunov's direct method can be extended to provide exponential stability statements. Specifically, the last condition on V in Theorem 3 will have to be replaced by $\dot{V}(x) \leq -\alpha V(x)$, where $\alpha > 0$ would form an estimate for the convergence rate. The discussion on exponential stability is not pursued further here.

2.3 Summary

This chapter provided a family of stability definitions that allow classifying equilibria in terms of their stability properties. We distinguished between stability, asymptotic stability and exponential stability. Exponential stability was introduced as a notion but was not discussed further. To this end, let $x^\star$ denote an equilibrium point of a system $\dot{x}(t) = f(x(t))$, with $x(0) = x_0$. The notion of stability and asymptotic stability properties of $x^\star$ can be then summarized as follows.

Stability

An equilibrium $x^\star$ is called stable if for any $\epsilon > 0$, there exists $\delta > 0$, such that

$$\|x_0 - x^\star\| < \delta \implies \|x(t) - x^\star\| < \epsilon, \text{ for all } t \geq 0.$$

Asymptotic stability

An equilibrium $x^\star$ is called locally asymptotically stable if

1. it is stable;
2. and there exists $\delta > 0$, such that

$$\|x_0 - x^\star\| < \delta \implies \lim_{t\to\infty} x(t) = x^\star.$$

An equilibrium is globally asymptotically stable if this holds for any $\delta > 0$.

We have discussed Lyapunov's direct and indirect methods to decide on the stability properties of an equilibrium point. The main results obtained are summarized below.

Lyapunov's indirect method (Theorem 1)
Consider the system $\dot{x}(t) = f(x(t))$ and assume f is continuously differentiable. Consider also its linearization around an equilibrium $x^\star$ with state-space matrix A. The equilibrium $x^\star$ is:

1. Locally asymptotically stable if all eigenvalues of A have negative real part.
2. Unstable if A has at least one eigenvalue with positive real part.

Lyapunov's direct method
Let $x^\star$ be an equilibrium of $\dot{x}(t) = f(x(t))$. Assume that there exists an open set $S \subseteq \mathbb{R}^n$ that contains the equilibrium, i.e., $x^\star \in S$. Assume also that there exists a continuously differentiable function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ such that

1. $V(x^\star) = 0$,
2. $V(x) > 0$, for all $x \in S$ with $x \neq x^\star$,
3. **Case A.** If in addition to items 1 & 2,

$$\dot{V}(x) \leq 0, \text{ for all } x \in S,$$

then the equilibrium $x^\star$ is stable. (**Theorem 2**)
Case B. If in addition to items 1 & 2,

$$\dot{V}(x) < 0, \text{ for all } x \in S \text{ with } x \neq x^\star,$$

then the equilibrium $x^\star$ is locally asymptotically stable. If $S = \mathbb{R}^n$ then the equilibrium point $x^\star$ is globally asymptotically stable. (**Theorem 3**)

3 Invariance & extensions to time-varying systems

In the previous chapter we provided tools to assess the stability properties of autonomous nonlinear systems. The strongest of these results involved a candidate Lyapunov function $V(x)$ whose time derivative was strictly decreasing along state trajectories, i.e., $\dot{V}(x) < 0$, for all points in the underlying domain except the equilibrium. Under this decrease condition, asymptotic stability (local or global) of the equilibrium point could be ensured. However, it is often difficult to construct a Lyapunov function with $\dot{V}(x) < 0$ on all points of a given domain; the pendulum example provides an instance when this fails to be the case.

In this chapter we consider the case where we only have a Lyapunov-like function with $\dot{V}(x) \leq 0$, which is more likely to occur. We aim at addressing the following question: If we only have $\dot{V}(x) \leq 0$ can we still comment on whether state trajectories $x(t)$ converge to some set as time tends to infinity?

1. We first provide an answer to this question for autonomous systems. This result complements asymptotic stability as it shows convergence to a set rather than a point, however, without requiring $\dot{V}$ to be strictly negative on the entire domain of interest. Moreover, we show that the set where trajectories converge is itself invariant, thus our analysis provides an invariance construction. This reasoning is known as La Salle's invariance principle.
2. We then extend our analysis to provide a generalization of La Salle's invariance principle to time-varying systems, and characterize the set towards which state trajectories converge. To deal with time-varying systems we first extend some of the stability concepts to time-varying systems, and then analyze the convergence properties of their state solutions.

3.1 Invariance

We first provide an invariant set construction: we show that when trajectories start from an invariant set, and under certain conditions, trajectories converge to yet another invariant set. This is formalized in the following theorem.

Theorem 4 (La Salle's Invariance Principle (Local)). *Let $x^\star$ be an equilibrium of $\dot{x}(t) = f(x(t))$. Assume that there exists a compact and invariant set $S \subseteq \mathbb{R}^n$ that contains the equilibrium, i.e., $x^\star \in S$. Assume also that there exists a continuously differentiable function $V(\cdot): \mathbb{R}^n \to \mathbb{R}$ such that*

$$\dot{V}(x) = \nabla V(x) f(x) \leq 0, \text{ for all } x \in S.$$

Denote by $\overline{S}$ the largest invariant set contained in

$$\Omega = \{x \in S : \ \dot{V}(x) = \nabla V(x) f(x) = 0\}.$$

We then have that every solution $x(t)$ with $x_0 \in S$, converges to $\overline{S}$ as $t \to \infty$.

The following remarks are in order:

1. If we have access to a candidate Lyapunov function V (see properties this needs to satisfy in Theorem 2), then we can construct an invariant set by looking at any of its level-sets. To this end, for any $c > 0$, consider the set

$$S_c = \{x \in \mathbb{R}^n : \ V(x) \leq c\}.$$

   This set is invariant due to the non-increase condition of a Lyapunov function: if we start at a given level set, we cannot move to higher level sets along the system trajectories. As such, any trajectory starting in S_c will stay in that set for all future time instances. However, it does not have to converge to $x^\star$ as V only satisfies the conditions for stability ($\dot{V}(x) \leq 0$), and not asymptotic stability. Moreover, this set is also compact (it does not extend to infinity and its complement is open) provided that from any direction that x tends to infinity, V tends to infinity as well, i.e., $V(x) \to \infty$ as $|x| \to \infty$; such a V is termed radially unbounded.

2. If an equilibrium $x^\star$ happens to be the only invariant set (recall that an equilibrium is itself an invariant set) contained in the set Ω of Theorem 4, then all trajectories starting in S will remain bounded (as S is assumed to be compact) and converge to $x^\star$. In other words, $\overline{S} = \{x^\star\}$. This would then imply that $x^\star$ is an asymptotically stable equilibrium. As such, La Salle's principle provides then (in some cases) alternative means to provide asymptotic stability assertions.

3. The sets involved in Theorem 4 satisfy the following inclusion

$$\overline{S} \subseteq \Omega \subseteq S,$$

while if $S = S_c$ as per the first remark above, we have $\overline{S} \subseteq \Omega \subseteq S_c$. Notice that S is invariant by construction, and $\overline{S}$ is invariant by the theorem's statement. The set Ω is not necessarily invariant.

We illustrate all these remarks in the example below.

Example 7. *Consider the setting of Example 6 with $d > 0$, and the associated Lyapunov function V. Use La Salle's invariance principle to show that $\overline{S} = \{x^\star\}$, where $x^\star = (0, 0)$. Show then that $x^\star$ is an asymptotically stable equilibrium.*

Solution: *We proceed in three main steps, constructing the sets S, Ω, and $\overline{S}$ in Theorem 4 and the remarks thereafter.*

1. *Set S: Consider the Lyapunov function of Example 6. Based on remark 1 above we take (notice this is compact given that $x_1 \in (-\pi, \pi)$)*

$$S = S_c = \{x \in \mathbb{R}^n : \ V(x) \leq c\}, \text{ with } c < 2mgl.$$

   Notice that S_c is invariant due to the fact that V is non-increasing. We choose $c < 2mgl$, as this corresponds to the energy the pendulum would have at the upside down vertical position with $x_1 = \pi$, $x_2 = 0$. The inequality is strict so that we do not include the other equilibrium $(\pi, 0)$.

2. *Set Ω: We have $\dot{V}(x) = \nabla V(x) f(x) = -dlx_2^2$ (see computation in Example 6). Notice that $\dot{V}(x) \leq 0$, for all $x \in S$, as V has been shown to be a Lyapunov function. Moreover, $\dot{V}(x) = 0$ for all points in S with $x_2 = 0$, while $\dot{V}(x) < 0$ for any other point in S ($d > 0$). We thus have that*

$$\Omega = \{x \in S : \ \nabla V(x) f(x) = 0\} = \{x \in S : \ x_2 = 0\}.$$

   Notice that $\Omega \subseteq S$ as it contains all points in S for which $x_2 = 0$.

3. *Set $\overline{S}$: The only invariant set in Ω is the origin. To see this, notice that for the state to remain in $x_2 = 0$ we must have that $\dot{x}_2(t) = 0$, as otherwise*

x_2 would move away from zero. This then implies that we must have $-\frac{g}{l}\sin x_1(t) = 0$, which since we are in S leads to $x_1 = 0$. Therefore,

$$\overline{S} = \{x^\star\} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

Theorem 4 implies then that all trajectories starting in S converge to $x^\star$, which by means of remark 2 above implies that $x^\star$ is locally asymptotically stable. Note that we had already shown by means of Lyapunov's direct method in Example 6 that $x^\star$ is stable; here we provide a stronger statement by means of La Salle's invariance principle. Moreover, by the choice of c in S_c, the domain of attraction of $x^\star$ contains all points in the c-level set of V, thus excluding the other equilibrium $(\pi, 0)$.
The constructed sets are also illustrated in Figure 10: $S = S_c$ is indicated by white with black boundary; Ω is the solid blue interval along the x_1 axis with $x_2 = 0$; and, $\overline{S}$ is the origin. Notice that inline with remark 3, $\overline{S} \subseteq \Omega \subseteq S_c$.

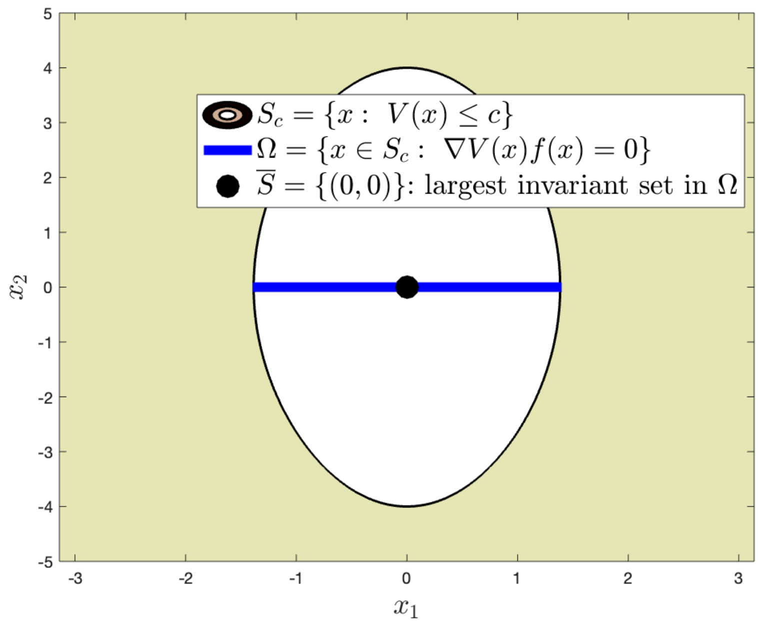

Figure 10: Invariant set construction for the application of La Salle's theorem to show convergence to the equilibrium point (origin). Notice that the depicted sets satisfy $\overline{S} \subseteq \Omega \subseteq S_c$.

Note that the La Salle's invariance principle in Theorem 4 is referred to as local, as convergence to $\overline{S}$ is guaranteed only if trajectories start in S which might not be the entire state space.

3.2 Extensions to time-varying systems

We now provide a generalization of La Salle's invariance principle to time-varying nonlinear systems. To this end, we first extend some of the stability results developed for autonomous systems to their time-varying counterparts. We only do this for stability definitions. Asymptotic and exponential stability definitions can be extended analogously, however, this is not pursued further in these notes. As we focus now on time-varying systems, we consider for the results of this section that t_0 is not necessarily zero, but arbitrary.

3.2.1 Stability

Consider a time-varying, nonlinear system of the form

$$\dot{x}(t) = f(x(t), t),$$

and denote an equilibrium point of this system by $x^\star \in \mathbb{R}^n$. Note that the definition of an equilibrium for time-varying systems is identical to that of autonomous ones. We then have the following generalization of the definition of stability to time-varying systems.

An equilibrium $x^\star$ of a time-varying nonlinear system is called stable if for any time t_0, and for any $\epsilon > 0$, there exists $\delta(t_0) > 0$ such that

$$\|x(t_0) - x^\star\| < \delta(t_0) \Longrightarrow \|x(t) - x^\star\| < \epsilon, \text{ for all } t \geq t_0.$$

It is called uniformly stable if δ is independent of t_0. An equilibrium is called unstable if it is *not* stable.

If a system is autonomous, then a stable equilibrium point is necessarily a uniformly stable one. This is since the state $x(t)$, $t \geq t_0$, of an autonomous system depends on $x(t_0)$ but not explicitly on t_0. Therefore, unlike autonomous systems, the constant δ in the definition of stability is a function of the initial time t_0.

Note that stability (as opposed to uniform stability) is a rather weak notion. For a fixed $\epsilon > 0$, $\delta(t_0)$ tends to zero as t_0 increases. This implies that the set of states around an equilibrium from which system trajectories can start and remain within

ϵ from $x^\star$ for all future times becomes progressively smaller. If this is not the case, and there is uniform lower bound on $\delta(t_0)$, then the system would be uniformly stable. As such, (non-uniform) stability implies that an equilibrium points tends to instability as $t_0 \to \infty$ (as $\delta(t_0)$ tends to zero).

We will now present the generalization of Lyapunov's direct method to investigate stability of time-varying systems. We state the theorem so that $x^\star = 0$ is a stable equilibrium point. This is without loss of generality, as through the coordinate change in p. 11, we can always have the origin as an equilibrium point. Moreover, we need the following definitions.

Positive definite time-invariant function. A time-invariant continuous function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is said to be positive definite on an open set $S \subseteq \mathbb{R}^n$, if $V(0) = 0$ and $V(x) > 0$ for any $x \in S$ with $x \neq 0$.

Positive definite time-varying function. A time-varying continuous function $V(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is said to be positive definite on an open set $S \subseteq \mathbb{R}^n$, if there exists a time-invariant positive definite function $V_L(x)$ on S such that

$$V(x,t) \geq V_L(x), \text{ for all } t \geq t_0, \ x \in S,$$

i.e., $V(x,t)$ is lower-bounded by V_L for all admissible time instances and states.

Decrescent function. A time-varying continuous function $V(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is said to be decrescent on an open set $S \subseteq \mathbb{R}^n$, if there exists a time-invariant positive definite function $V_U(x)$ on S such that

$$V(x,t) \leq V_U(x), \text{ for all } t \geq t_0, \ x \in S,$$

i.e., $V(x,t)$ is upper-bounded by V_U for all admissible time instances and states.

We say that V is negative definite if $-V$ is positive definite. If $S = \mathbb{R}^n$ these definitions become global as opposed to local. For a time-varying function, being positive definite or decrescent encodes a dominance condition (from below or above, respectively) by a time-invariant positive definite function.

As example, let $t_0 = 0$. Consider the function $V(x,t) = (t+1)\|x\|^2 \geq \|x\|^2$. Notice that the inequality holds for all $t \geq 0$. As such this a positive definite function under the choice $V_L(x) = \|x\|^2$. Similarly, consider the function $V(x,t) = e^{-t}\|x\|^2 \leq$

$\|x\|^2$, where the inequality holds for all $t \geq 0$. As such this is a decrescent function under the choice $V_U(x) = \|x\|^2$.

Theorem 5 (Lyapunov's direct method – Stability (Time-Varying Systems)). *Let $x^\star = 0$ be an equilibrium of $\dot{x}(t) = f(x(t), t)$. Assume that there exists an open set $S \subseteq \mathbb{R}^n$ that contains the equilibrium , i.e., $x^\star \in S$. Assume also that there exists a continuously differentiable function $V(\cdot,\cdot): \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that*

1. *$V(x,t)$ is positive definite on S,*
2. *$\dot{V}(x,t) = \frac{\partial V}{\partial t}(x,t) + \frac{\partial V}{\partial x}(x,t) f(x) \leq 0$, for all $x \in S$.*

We then have that the equilibrium point $x^\star = 0$ is stable.
If in addition to items 1 & 2 we have that $V(x,t)$ is decrescent on S, then the equilibrium point $x^\star = 0$ is uniformly stable.

Notice that the time derivative of V now includes the term $\frac{\partial V}{\partial t}(x,t)$, as time appears as an explicit argument of V. The proof follows similar arguments with that of the autonomous case, with modifications to account for the time-dependency of V. In particular, we replace $V(x)$ in the proof of Theorem 2 with $\min_{t \geq t_0} V(x,t)$; the proofline remains then the same.

3.2.2 Convergence

We now generalize La Salle's invariance principle to time-varying systems. In particular, we will provide the time-varying counterpart of Theorem 4 characterizing the set of states towards which state trajectories converge. We first present the following auxiliary result that we will use in the sequel, when providing a proof for the generalization of La Salle's invariance principle.

Barbalat's Lemma. If a function $\phi(\cdot): \mathbb{R} \to \mathbb{R}$ is *uniformly continuous*, and in addition its integral exists and is finite, i.e.,

$$\int_0^\infty \phi(t)\, \mathrm{d}t < \infty,$$

then $\lim_{t\to\infty} \phi(t) = 0$.

Note that uniform continuity is a stronger notion than continuity; see Appendix 5.1 for a definition and examples. It should also be noted that not all functions whose integral is finite satisfy $\lim_{t\to\infty}\phi(t) = 0$. As an example, consider the function of Figure 11. This is a train of triangle functions, each of them with height one and area $1/i!$, $i = 0, 1, \ldots$. Hence, its integral is finite as $\int_0^\infty \phi(t)dt = \sum_{i=0}^\infty \frac{1}{i!} = e$. However, $\phi(t)$ does not tend to zero as $t \to \infty$. The reason is that this function is not uniformly continuous as it tends to be discontinuous as $t \to \infty$ (more formally notice that there is no non-zero $\delta > 0$ that satisfies the uniform continuity definition).

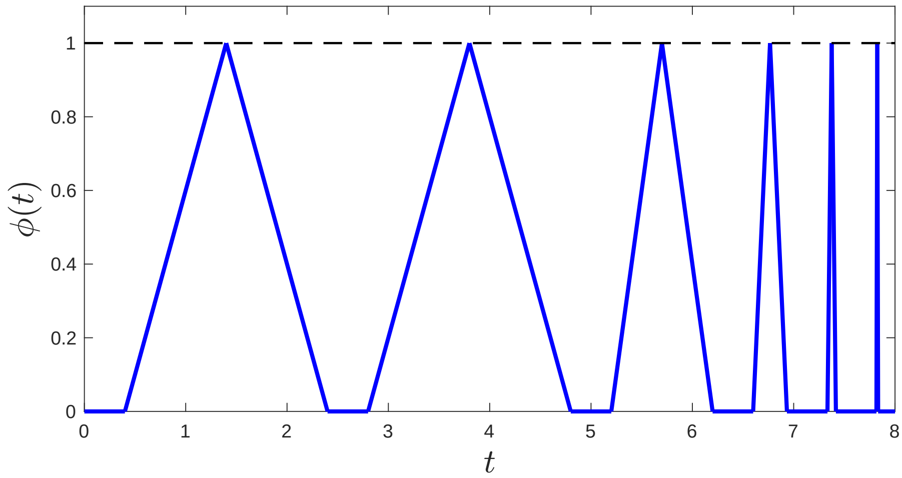

Figure 11: Train of triangle functions, each of them with height one and area $1/i!$, $i = 0, 1, \ldots$. This is an example of a function $\phi(t)$ that does not tend to zero as $t \to \infty$, however, its integral is finite, namely, $\int_0^\infty \phi(t)dt = \sum_{i=0}^\infty \frac{1}{i!} = e$.

For a given scalar r denote by $B_r = \{x \in \mathbb{R}^n : \ \|x\| < r\}$ and $\overline{B}_r = \{x \in \mathbb{R}^n : \ \|x\| \leq r\}$ an open and closed "ball", respectively, of radius r. Also, we say that a continuous function $W(x)$ defined on some domain S is non-negative, if $W(0) = 0$ and $W(x) \geq 0$, for all $x \in S$. Notice that this is a more relaxed property than positive definiteness as we allow W to be zero also at other points besides the origin.

We are now ready to provide a generalization of La Salle's invariance principle to time-varying systems. Note that, similarly to Theorem 5, we assume without loss of generality that $x^\star = 0$ is an equilibrium of the system.

Theorem 6 (Generalization of La Salle's Invariance Principle (Local)). *Let $x^\star = 0$ be an equilibrium of $\dot{x}(t) = f(x(t), t)$. Assume that there exists an*

open set $S \subseteq \mathbb{R}^n$ that contains the equilibrium , i.e., $x^\star \in S$.
Consider r such that $B_r \subset \overline{B}_r \subset S$. Assume that f is Lipschitz continuous in x uniformly with respect to t, on B_r. Assume also that there exists a continuously differentiable function $V(\cdot,\cdot): \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that

1. *$V(x,t)$ is positive definite and decrescent on S,*
2. *for some non-negative function $W(x)$ on S, we have that*

$$\dot{V}(x,t) = \frac{\partial V}{\partial t}(x,t) + \frac{\partial V}{\partial x}(x,t) f(x) \leq -W(x) \leq 0, \text{ for all } x \in S.$$

We then have that there exists a set of initial states $X_0 \subseteq B_r$ such that all trajectories $x(\cdot)$ that start therein remain bounded, and

$$\lim_{t\to\infty} W(x(t)) = 0.$$

Proof. Since $V(x,t)$ is positive definite and decrescent on S, there exist positive definite functions V_L and V_U such that $V_L(x) \leq V(x,t) \leq V_U(x)$, for all all $t \geq t_0$, and all $x \in S$. Consider the set

$$X_0 = \{x \in B_r: \ V_U(x) < \min_{\{x:\ \|x\|=r\}} V_L(x)\}.$$

We will show that any trajectory with initial state $x(t_0) \in X_0$ remains bounded. For the sake of contradiction assume that this is not the case, and there exists $x_0 = x(t_0) \in X_0$ such that the trajectory that starts from that state escapes to infinity. If this is the case, there would exist some time T such that the state trajectory escapes B_r and touches its boundary, i.e., $\|x(T)\| = r$. We have that

$$\begin{aligned} V_L(x(T)) &\leq V(x(T),T) \\ &\leq V(x_0,t_0) \leq V_U(x_0) < \min_{\{x:\ \|x\|=r\}} V_L(x) \leq V_L(x(T)), \end{aligned}$$

where the first inequality is since $V(x,t)$ is positive definite, the second one is due to the fact that $\dot{V}(x,t) \leq 0$ along the state trajectories, and the third one is since $V(x,t)$ is decrescent. The strict inequality is since we have assumed that $x_0 \in X_0$, while the last inequality is since $\min_{\{x:\ \|x\|=r\}} V_L(x) \leq V_L(x(T))$, as $\|x(T)\| = r$. This establishes a contradiction (as $V_L(x(T)) < V_L(x(T))$). We have thus shown that all trajectories that start in X_0 remain bounded (in fact they stay within B_r).

It remains to show that $\lim_{t\to\infty} W(x(t)) = 0$. To show this we will be invoking Barbalat's lemma. To this end, notice that

$$\int_{t_0}^{\infty} W(x(t))\ \mathrm{d}t \leq \int_{t_0}^{\infty} -\dot{V}(x(t),t)\ \mathrm{d}t = V(x_0,t_0) - \lim_{t\to\infty} V(x(t),t),$$

where the inequality is since $\dot{V}(x(t),t) \leq -W(x)$, and the equality is by the definition of a definite integral. Notice that since $V(x,t)$ is positive definite, $\lim_{t\to\infty} V(x(t),t) \geq 0$, while $V(x_0,t_0)$ is finite. As such, we have that

$$\int_{t_0}^{\infty} W(x(t))\ \mathrm{d}t < \infty \implies \int_{0}^{\infty} W(x(t))\ \mathrm{d}t < \infty,$$

where the implication is since the integral of W (recall it is continuous as it is non-negative) over $[0,t_0]$ is finite. At the same time we have that:

1. The state $x(t)$ is a uniformly continuous function of t. This follows by the discussion in Appendix 5.2, since the state is bounded on B_r, and f is Lipschitz continuous in x uniformly with respect to t (i.e., the Lipschitz constant is independent of t) over the same domain.
2. $W(x)$ is a uniformly continuous function of x over $\overline{B}_r \subset S$. This follows since W is continuous (as it is non-negative) over S, and $\overline{B}_r$ is compact. Therefore, continuity over a compact set (see first bullet point at the end of Appendix 5.1) implies that W is also uniformly continuous over $\overline{B}_r$.
3. $W(x(t))$ is a uniformly continuous function of t. This follows since composition of uniformly continuous functions preserves uniform continuity; see second bullet point at the end of Appendix 5.1, and notice that $S_g = B_r$ is the codomain of $x(\cdot)$, while $S_f = \overline{B}_r$ is the domain of W, and they satisfy $S_g \subset S_f$.

By the previous arguments we have that $\phi(\cdot) = W(x(\cdot)) :\ \mathbb{R} \to \mathbb{R}$ is uniformly continuous and its integral was shown to be finite. As such, by Barbalat's lemma

$$\lim_{t\to\infty} W(x(t)) = 0,$$

thus concluding the proof. □

An equivalent way of stating the final conclusion of Theorem 6 is to say that as $t \to \infty$, the sate $x(t)$ approaches the set $\{x \in B_r :\ W(x) = 0\}$. However, this set is not guaranteed to be invariant, neither we can assert that $x(t)$ tends to the largest invariant set inside that set. Such a conclusion would be valid if the system was autonomous, or if the vector field was periodic in time (termed T-periodic).

3.3 Summary

In this chapter we investigated the case where a candidate Lyapunov function has a time derivative that is *not* strictly decreasing along the system's trajectories. In particular, we characterize the set of states towards which trajectories converge as time tends to infinity. We first focused on the autonomous case; our main result in this direction is summarized into La Salle's invariance principle.

La Salle's Invariance Principle (Theorem 4)
Let $x^\star$ be an equilibrium of $\dot{x}(t) = f(x(t))$. Assume that there exists a compact and invariant set $S \subseteq \mathbb{R}^n$ that contains the equilibrium, i.e., $x^\star \in S$. Assume also that there exists a continuously differentiable function $V(\cdot): \mathbb{R}^n \to \mathbb{R}$ such that

$$\dot{V}(x) = \nabla V(x) f(x) \leq 0, \text{ for all } x \in S.$$

Denote by $\overline{S}$ the largest invariant set contained in

$$\Omega = \{x \in S : \dot{V}(x) = \nabla V(x) f(x) = 0\}.$$

We then have that every solution $x(t)$ with $x_0 \in S$, converges to $\overline{S}$ as $t \to \infty$.

We then addressed the same question for time-varying nonlinear systems. For these developments we considered without loss of generality that $x^\star = 0$ is an equilibrium of the system. We extended Lyapunov's direct method to time varying systems (see Theorem 5); to this end, we introduced the notion of positive definite and decrescent functions. In particular, if a function $V(x,t)$ is both positive definite and decrescent on a set S, then there exist functions $V_L(x)$ and $V_U(x)$ (positive definite time invariant ones, i.e., they are zero at the origin and positive for any other admissible x), such that

$$V_L(x) \leq V(x,t) \leq V_U(x), \text{ for all all } t \geq t_0, \text{ and all } x \in S.$$

We then established the following generalization of La Salle's invariance principle.

Generalization of La Salle's Invariance Principle (**Theorem 6**)

Let $x^\star = 0$ be an equilibrium of $\dot{x}(t) = f(x(t), t)$. Assume that there exists an open set $S \subseteq \mathbb{R}^n$ that contains the equilibrium , i.e., $x^\star \in S$.

Assume that f is Lipschitz continuous in x uniformly with respect to t, on an open "ball" of radius r (with the closed ball contained in S). Assume also that there exists a continuously differentiable function $V(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that

1. $V(x, t)$ is positive definite and decrescent on S,
2. for some non-negative function $W(x)$ on S, we have that

$$\dot{V}(x,t) = \frac{\partial V}{\partial t}(x,t) + \frac{\partial V}{\partial x}(x,t) f(x) \leq -W(x) \leq 0, \text{ for all } x \in S.$$

We have that there exists a set of initial states X_0 (subset of the "ball" of radius r) such that all trajectories $x(\cdot)$ that start therein remain bounded, and

$$\lim_{t \to \infty} W(x(t)) = 0.$$

4 Linear & passive systems

In the previous chapters we have provided different ways to decide on the stability of nonlinear systems, most importantly, Lyapunov direct methods. We have also discussed approaches on establishing convergence of state trajectories to some set of states, establishing La Salle's invariance principle and its generalization to time-varying systems. These results rely on the availability of a Lyapunov-like function that satisfies certain conditions. In general, constructing a Lyapunov function is a difficult task and requires experience. However, for certain classes of systems more systematic approaches exist. To this end, in this chapter:

1. We show how our stability results specialize to the class of linear systems. This offers additional insights, but at the same time it also provides alternative means to check stability of linear systems. In particular, rather than directly computing the eigenvalues of the associated state space matrix, Lyapunov's stability theory involves solving a system of linear equations to reach to the same stability conclusion.
2. We introduce and analyze the class of the so called passive systems, that admit an energy-related interpretation, as well as feedback interconnections of them. We discuss the implications of passivity in terms of the stability properties of autonomous nonlinear systems. Specializing to linear systems we provide the so called Kalman-Yakubovich-Popov lemma that provides a systematic approach to decide on the passivity properties of the underlying system.
3. We investigate the stability properties of feedback control systems that involve a linear plant (that enjoys certain passivity properties) and a nonlinear controller (of specific type). We derive the so called circle criterion, which provides an extension of the Nyquist stability criterion to nonlinear systems of this form.

4.1 Linear systems

Consider a linear time-invariant (LTI) system with no inputs, hence, autonomous, represented by

$$\dot{x}(t) = Ax(t),$$

where $A \in \mathbb{R}^{n\times n}$ is the state space matrix. Notice that $x^\star = 0$ is an equilibrium of the system; we will investigate the stability properties of this equilibrium point. To this end, recall that the origin of an autonomous LTI system is asymptotically stable if and only if all eigenvalues of A have negative real part.

Here we will establish an alternative necessary and sufficient condition for asymptotic stability using Lyapunov's direct method specialized to LTI systems. It turns out that for LTI systems a quadratic function becomes a viable choice for a Lyapunov function. To this end, let $P = P^\top \succ 0$ be a symmetric positive definite matrix and consider the following quadratic candidate Lyapunov function

$$V(x) = x^\top P x.$$

Recall that we say that P is positive definite, and write $P \succ 0$, if $x^\top P x > 0$ for all $x \neq 0$. If these inequalities are non-strict, then we say that the underlying matrix is positive semidefinite. Notice that a positive definite matrix gives rise to a positive definite function V according to the definitions provided in the previous chapter. Note also that considering P to be symmetric is without loss of generality as for non-symmetric matrices the skew-symmetric part vanishes within a quadratic[§].

Considering the conditions a candidate Lyapunov function needs to satisfy, notice that for $V(x) = x^\top P x$ and $x^\star = 0$, we have that

1. $V(0) = 0$;
2. $V(x) > 0$, for all $x \neq 0$;
3. Taking the time derivative of V across state trajectories, we obtain that

$$\begin{aligned}\dot V(x) &= \dot x^\top P x + x^\top P \dot x \\ &= x^\top (A^\top P + PA) x = -x^\top Q x,\end{aligned}$$

   where in the last step we introduced a matrix Q such that $A^\top P + PA = -Q$ (notice that Q is symmetric). This is known as the Lyapunov matrix equation.

[§]To see this, assume that P is not symmetric, and hence $P \neq P^\top$. Notice then that

$$x^\top P x = \frac{1}{2} x^\top (P + P^\top) x + \frac{1}{2} x^\top (P - P^\top) x = \frac{1}{2} x^\top (P + P^\top) x,$$

where the last equality is since $x^\top P x = x^\top P^\top x$, as they are scalar quantities and hence $(x^\top P^\top x)^\top = x^\top P x$. Therefore, if a non-symmetric matrix appears in a quadratic expression, then only the symmetric part $\frac{1}{2}(P + P^\top)$ of the matrix has a non-zero contribution and the skew-symmetric part vanishes.

If Q is positive definite, then $\dot{V}(x) < 0$ so we can directly apply Lyapunov's direct method in Theorem 3 and infer that $x^\star = 0$ is an asymptotically stable equilibrium, and hence that all eigenvalues of A have negative real part. Notice that we can take $S = \mathbb{R}^n$ in the theorem's statement, hence the conclusion would be global.

However, for an arbitrary $P = P^\top \succ 0$, the resulting Q may not necessarily be positive definite, thus not allowing us to inherit the desired stability conclusion. Therefore, we revert the oder in the choice of P and Q. Start first by choosing a symmetric positive definite matrix $Q = Q^\top \succ 0$, and then determine a matrix $P = P^\top \succ 0$ that satisfies the Lyapunov's matrix equation for the given Q. If such a P exists then the resulting quadratic $x^\top P x$ would be a valid Lyapunov function, and by Theorem 3 we can still assert that the origin is an aymptotically stable equilibrium. It turns out that for systems where all eigenvalues of A have negative real part (asymptotically stable LTI stystems), given a matrix Q there always exists a P satisfying the Lyapunov matrix equation, and in fact this is a unique one.

Theorem 7 (Lyapunov Stability of (Autonomous) LTI Systems)**.** *Consider an LTI system $\dot{x}(t) = Ax(t)$. The system is asymptotically stable if and only if for any given $Q = Q^\top \succ 0$, there exists a unique $P = P^\top \succ 0$ satisfying the Lyapunov matrix equation*

$$A^\top P + PA = -Q.$$

To gain some further insights on Theorem 7, note that if the system is asymptotically stable then, given any symmetric positive definite Q, the unique symmetric positive definite matrix P that satisfies the Lyapunov matrix equation is given by

$$P = \int_0^\infty e^{A^\top t} Q e^{At} \, \mathrm{d}t.$$

We will not show uniqueness, but we will show that indeed this particular choice satisfies $P = P^\top \succ 0$ and $A^\top P + PA = -Q$. Notice first that the integral exists; this is since the system is taken to be asymptotically stable, hence e^{At} will involve terms that will be dominated by decaying exponentials with the exponents being the real parts of the eigenvalues of A, which are all negative. As such, the integral will be finite. Moreover, since $Q = Q^\top \succ 0$, we will also have by construction that $P = P^\top \succ 0$. Substituting now our choice for P in the Lyapunov matrix equation, we obtain

$$\begin{aligned} A^\top P + PA &= \int_0^\infty A^\top e^{A^\top t} Q e^{At}\,\mathrm{d}t + \int_0^\infty e^{A^\top t} Q e^{At} A\,\mathrm{d}t \\ &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} e^{A^\top t} Q e^{At}\,\mathrm{d}t = e^{A^\top t} Q e^{At}\Big|_0^\infty = -Q, \end{aligned}$$

where the second equality is by noticing that the sum of the integrands in the right-hand side of the first equality is the total derivative of $e^{A^\top t} Q e^{At}$ with respect to t, while as $t \to \infty$, $e^{A^\top t} Q e^{At}$ tends to zero following the previous argumentation on the integral existence, since A has eigenvalues with negative real part.

The Lyapunov matrix equation could be written as a system of linear equations with the unknowns being the elements of matrix P (which could be rearranged in a column vector format). Theorem 7 suggests that solving this linear system constitutes an alternative way of determining whether an autonomous LTI system is asymptotically stable, compared to computing the eigenvalues of the A matrix and checking whether they have negative real parts. From a computation point of view, however, this is not necessarily easier as numerical methods typically used to solve that system provide as a byproduct the eigenvalues of A. However, the main advantage of Theorem 7 is that it offers a procedure of finding a Lyapunov function for autonomous LTI systems. In particular, it suggests solving the Lyapunov matrix equation, and then constructing a quadratic Lyapunov function $V(x) = x^\top P x$ with P being that solution. Having access to a Lyapunov function would be of use when considering linear systems with inputs as in the subsequent subsections.

4.2 Passive systems

Passive nonlinear systems constitute an important class of systems, as their analysis in terms of stability becomes more systematic and the construction of Lyapunov-like functions admits naturally an energy interpretation. Here we will consider the case where the number of inputs is equal to the number of outputs. To relate passivity to physical systems, consider a (possibly nonlinear) electric circuit as in Figure 12, including storage elements like inductors and capacitors and possibly also resistive ones. The inputs are independent voltage and current sources, u_1 and u_2, respectively, with reference directions as in the figure. The quantity $u^\top y = u_1 y_1 + u_2 y_2$ has the interpretation of the instantaneous power input into the circuit.

Informally, the system is said to be passive if the amount of input energy (the integral of power) is greater than or equal to the change in the amount of energy stored in the circuit. To this end, let $V(x)$ denote the energy stored in the circuit, as a function of the state vector x, e.g., the voltage at capacitors, current at inductors, etc. Passivity as described above would then take the form of

$$\int_0^t u(\tau)^\top y(\tau)\ \mathrm{d}\tau \geq V(x(t)) - V(x(0)),$$

where if the inequality is strict, the energy has to be dissipated across resistive elements. If this inequality holds for any $t \geq 0$ then, by differentiation, we obtain the following inequality for the instantaneous power

$$u(t)^\top y(t) \geq \dot{V}(x(t)), \text{ for all } t \geq 0.$$

This inequality captures the notion of passivity, and will be formalized next.

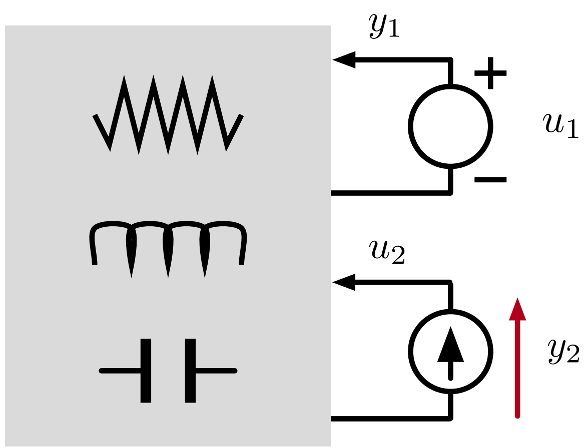

Figure 12: Pictorial illustration of a passive electric circuit, including resistors, inductors, and capacitors.

4.2.1 Passivity and stability

Consider the following time-invariant nonlinear system in state-space form.

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)),\\ y(t) &= h(x(t), u(t)),\end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$, with $p = m$ as we consider systems that have the same number of inputs and outputs. Assume without loss of generality that, in the absence of any input, the origin $x^\star = 0$ is an equilibrium point, i.e., $f(0,0) = 0$. To introduce the notion of passivity we first consider an example.

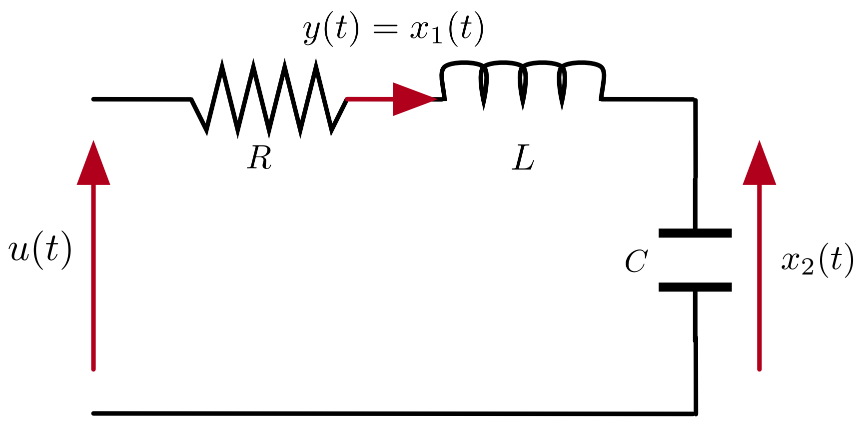

Figure 13: Electric circuit example.

Example 8. *Consider the RLC circuit of Figure 13. Let* $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$*, where* $x_1(t)$ *denotes the current across the inductor and* $x_2(t)$ *the voltage across the capacitor. Denote by*

$$V(x) = \frac{1}{2}Lx_1^2 + \frac{1}{2}Cx_2^2,$$

the energy stored in the circuit. Show that $u(t)y(t) \geq \dot{V}(x(t))$ *for all* $t \geq 0$.

Solution: *The voltage across the inductor is given by* $L\dot{x}_1(t)$*, and the current across the capacitor (equal to the current across the inductor as they are connected in series) is given by* $x_1(t) = C\dot{x}_2(t)$*. Moreover, by Kirchhoff's voltage law we have that* $u(t) = Rx_1(t) + L\dot{x}_1(t) + x_2(t)$ *(sum of voltages across the resistor, the inductor and the capacitor, respectively). Rearranging these equations and writing them in matrix format, we obtain the state space description of the system:*

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Consider the rate of change of the energy $V(x)$*, namely,*

$$\begin{aligned} \dot{V}(x(t)) &= Lx_1(t)\dot{x}_1(t) + Cx_2(t)\dot{x}_2(t) \\ &= -Rx_1^2(t) - \cancel{x_1(t)x_2(t)} + x_1(t)u(t) + \cancel{x_1(t)x_2(t)} \\ &\leq u(t)y(t), \end{aligned}$$

where the second equality is by substituting expressions for $\dot{x}_1(t)$ and $\dot{x}_2(t)$ based on the state space description, and the inequality is since $-Rx_1^2(t)$ is negative and $y(t) = x_1(t)$. This establishes the claim; in fact the inequality is strict as energy is dissipated on R. If there was only the resistor in the circuit, we could take $V(x) = 0$ and hence $\dot{V}(x) = 0$; we would then still have $u(t)y(t) = Ry^2(t) \geq 0 = \dot{V}(x(t))$ by Ohm's law.

We now provide a formal definition of passivity; to simplify notation we do not show t as the argument. Its presence will always be clear from the context.

The time-invariant nonlinear system under consideration is said to be

1. passive if there exists a continuously differentiable function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$, $V(x) \geq 0$ for all $x \in \mathbb{R}^n$, and

$$u^\top y \geq \dot{V}(x) = \nabla V(x) f(x,u), \text{ for all } (x,u) \in \mathbb{R}^n \times \mathbb{R}^m;$$

2. strictly passive if there exists a continuously differentiable function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$, $V(x) \geq 0$ for all $x \in \mathbb{R}^n$, as well as a positive definite function $W(\cdot) : \mathbb{R}^n \to \mathbb{R}$ such that

$$u^\top y - W(x) \geq \dot{V}(x) = \nabla V(x) f(x,u), \text{ for all } (x,u) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Strictly passive systems are often termed dissipative, while function V is often referred to as storage function. A positive definite function is zero at the origin and positive at all other points. Notice then the difference between V and W: the latter is only zero at the origin, while the former can also be zero at other points. The system in Example 8 dissipates energy; however, $W(x) = Rx_1^2$ can also be zero at points different from the origin. Such a system is termed output strictly passive, as such a W satisfies the desired properties as a function of the output only.

There seems to be a strong link between Lyapunov and storage functions. In fact, in the absence of inputs, storage functions satisfy a decrease condition; however, this is not enough to classify them as Lyapunov functions. The reason is that they are allowed to be zero even at points different from the equilibrium $x^\star = 0$. However, the proof of the following result (provided in Appendix 5.3), shows that under strict passivity, storage functions qualify as valid Lyapunov function candidates. As such, passivity offers an energy theoretic framework to construct Lyapunov functions and assess the stability properties of the autonomous system counterpart.

Theorem 8. *Let $x^\star = 0$ be an equilibrium of $\dot{x}(t) = f(x(t), 0)$. Assume that there exists an open set $S \subseteq \mathbb{R}^n$ that contains the equilibrium, i.e., $x^\star \in S$.*

- *If the system is strictly passive (V, W could be defined over the domain S rather than $\mathbb{R}^n$), then $x^\star = 0$ is an asymptotically stable equilibrium point.*

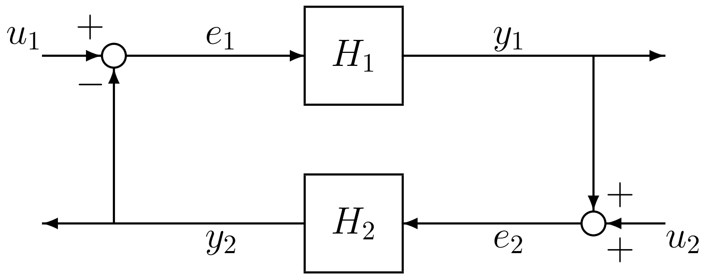

Figure 14: Feedback interconnection of passive systems.

Consider now the feedback interconnection of systems H_1 and H_2, as shown in Figure 14. These are given by (where dimensions are assumed to be consistent)

$$H_1: \; \dot{x}_1 = f_1(x_1, e_1), \qquad H_2: \; \dot{x}_2 = f_2(x_2, e_2),$$
$$y_1 = h_1(x_1, e_1). \qquad y_2 = h_2(x_2, e_2).$$

Using the expressions for y_1 and y_2, the junctions in Figure 14 lead to

$$e_1 = u_1 - y_2 = u_1 - h_2(x_2, e_2)$$
$$e_2 = u_2 + y_1 = u_2 + h_1(x_1, e_1).$$

Assume that for given x_1, x_2, u_1, u_2, there is a unique pair (e_1, e_2) satisfying these equations. This is for example the case if h_1 and h_2 are independent of e_1 and e_2, respectively (the input does not appear in the output). In such cases, the state solution of the closed loop system exists and is unique (provided this is also the case for H_1 and H_2). By the closed loop system here we mean the feedback interconnection with state $x = (x_1, x_2)$, input $u = (u_1, u_2)$, and output $y = (y_1, y_2)$.

If H_1 and H_2 are passive, then their feedback interconnection (closed loop system) as in Figure 14 is also passive.

To see this, assume H_1, H_2 are passive, and let V_1, V_2 be their respective storage functions. As such, $e_1^\top y_1 \geq \dot{V}_1(x_1)$ and $e_2^\top y_2 \geq \dot{V}_2(x_2)$. Take $V(x) = V_1(x_1) + V_2(x_2)$ as a candidate storage function for the closed loop system. Therefore,

$$\begin{aligned}\dot{V}(x) = \dot{V}_1(x_1) + \dot{V}_2(x_2) &\leq e_1^\top y_1 + e_2^\top y_2 \\ &= (u_1 - y_2)^\top y_1 + (u_2 + y_1)^\top y_2 \\ &= u_1^\top y_1 + u_2^\top y_2 = u^\top y,\end{aligned}$$

where the inequality is since V_1 and V_2 are storage functions for the respective systems. The second equality is since $e_1 = u_1 - y_2$ and $e_2 = u_2 + y_1$, and the third one since $y_1^\top y_2 = y_2^\top y_1$. Therefore, the closed loop system is also passive. Similar conclusions can be made for feedback interconnections of strictly passive systems.

4.2.2 Linear passive systems

We now investigate passivity when the underlying system is an LTI one, of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t),\end{aligned}$$

where $B \in \mathbb{R}^n$ and $C \in \mathbb{R}^{1\times n}$, i.e., we consider single-input, single-output LTI systems. Moreover, the input does not appear in the output equation, i.e., $D = 0$. These choices simplify some of the subsequent developments and can be relaxed.

Denote by $G(s) = C(sI - A)^{-1}B + D$ the system's transfer function. We introduce a frequency-domain notion that will be then related to passivity. We say that $G(s)$ is positive real if: (i) its poles have non-positive real part; (ii) $\mathrm{Re}[G(j\omega)] \geq 0$ for all $\omega \in \mathbb{R}$; (iii) if $j\omega$ is a pole of $G(s)$ then it is simple (of order one), and $\lim_{s\to j\omega}(s - j\omega)G(s) \geq 0$ holds[§] $G(s)$ is strictly positive real if there exists $\epsilon > 0$ such that $G(s - \epsilon)$ is positive real in a non-strict sense.

[§]This limit is a residue in complex analysis capturing the behaviour of $G(s)$ around a singularity on the imaginary axis, bypassed by an infinitesimal semi-circle on the right-half plane. For simple poles, in a Nyquist diagram this semi-circle maps through G to another semi-circle that closes Nyquist's curve through infinity via the right half-plane; for repeated poles this would require a full circular closure in the Nyquist diagram thus passing through the left-half plane. Note also that (i) implies that $G(s)$ is analytic (infinitely differentiable with convergent series expansion) on the open right-half plane, and hence $\mathrm{Re}[G(s)] \geq 0$ for all complex s on the open right-half plane, i.e., with $\mathrm{Re}[s] > 0$.

Property (ii) is instrumental in casting a given transfer function as positive real. It implies that the real part of its frequency response should be non-negative for any finite frequency. Property (i), requires all poles of $G(s)$ to have non-positive real part, thus allowing to check non-negativity of the steady state frequency response $G(j\omega)$. Singularities are addressed via (iii); see footnote, justifying why these conditions are present in the definition of positive real transfer functions. For our developments we are interested in the stronger notion of strictly positive realness; this can be checked by means of the following result without going through the definition of (non-strictly) positive real transfer functions.

The transfer function $G(s) = C(sI - A)^{-1}B + D \in \mathbb{C}$ is strictly positive real if and only if

1. all poles of $G(s)$ have negative real part;
2. $\mathrm{Re}[G(j\omega)] > 0$ for all $\omega \in \mathbb{R}$;
3. either $G(\infty) > 0$, or $G(\infty) = 0$ and $\lim_{\omega\to\infty} \omega^2 \mathrm{Re}[G(j\omega)] > 0$.

This results remains valid even if D is non-zero. Notice that the second condition requires the real part of $G(j\omega)$ (frequency response function) to be strictly positive for all finite $\omega \in \mathbb{R}$. Positivity in the limiting case $\omega \to \pm\infty$ is captured by the last condition, where by $G(\infty)$ we imply the limit of $G(j\omega)$ as $\omega \to \infty$. If $G(\infty) = 0$ (this is always the case when $D = 0$), then the real part of $G(j\omega)$ should not decay faster than $1/\omega^2$, as otherwise $\lim_{\omega\to\infty} \omega^2 \mathrm{Re}[G(j\omega)]$ would be zero. In the absence of the third condition, the first two become necessary but not sufficient.

Example 9. *Show that the transfer function:*

1. $G_1(s) = \frac{1}{s+a}$ *with* $a > 0$ *is strictly positive real;*
2. $G_2(s) = \frac{s+a+b}{(s+a)(s+b)}$ *with* $a, b > 0$ *is not strictly positive real.*

Solution:

1. $G_1(s)$ *has only one pole equal to* $-a < 0$. *Moreover,*

$$\mathrm{Re}[G_1(j\omega)] = \frac{a}{\omega^2 + a^2}, \text{ for all } \omega \in \mathbb{R}.$$

Since $a > 0$, $\mathrm{Re}[G_1(j\omega)] > 0$ for all $\omega \in \mathbb{R}$. Moreover, $G_1(\infty) = 0$ and

$$\lim_{\omega\to\infty} \omega^2 \mathrm{Re}[G_1(j\omega)] = \lim_{\omega\to\infty} \frac{a\omega^2}{\omega^2 + a^2} = a > 0.$$

As such, all three conditions in the definition of a positive real transfer function are satisfied. This then implies that $G_1(s)$ is strictly positive real. Its Nyquist plot[§] is shown in Figure 15. Notice that the real part is always strictly positive except at $\omega \to \infty$ (origin).

2. *$G_2(s)$ has two poles equal to $-a$ and $-b$, that are both negative. Moreover,*

$$\mathrm{Re}[G_2(j\omega)] = \mathrm{Re}\Big[\frac{j\omega + a + b}{(j\omega + a)(j\omega + b)}\Big] = \frac{ab(a+b)}{(\omega^2 + a^2)(\omega^2 + b^2)}, \text{ for all } \omega \in \mathbb{R}.$$

Since $a, b > 0$, $\mathrm{Re}[G_2(j\omega)] > 0$ for all $\omega \in \mathbb{R}$; however, $G_2(\infty) = 0$ but $\lim_{\omega\to\infty} \omega^2 \mathrm{Re}[G_2(j\omega)] = 0$. As such the third condition in the definition of a positive real transfer function is not satisfied. This then implies that $G_2(s)$ is not strictly positive real. Therefore, the first two conditions are not sufficient; $G_2(s)$ satisfies them but is not strictly positive real.

Its Nyquist plot is shown in Figure 15. Notice that as $\omega \to \infty$ the graph is flatter than that of $G_1(j\omega)$ as its decay rate is faster than $1/\omega^2$. Alternatively, for $G_2(s)$ to be strictly positive real, $G_2(s - \epsilon)$ should be positive real in a non-strict sense for some $\epsilon > 0$. Since $G_2(s)$ has negative real poles, this would mean that its Nyquist plot does not enter the left-half plane. However, for any $\epsilon > 0$ the locus of $G_2(s - \epsilon)$ enters the left-half plane; see Figure 15 for $\epsilon = 0.5$.

The following result links frequency domain properties of a system (encoded in the definition of strictly positive real transfer functions) to time domain algrebraic

[§]The Nyquist locus of a strictly positive real transfer function lies exclusively in the first and the fourth quadrants (see locus of $G_1(s)$ in Figure 15) since the real part of $G(j\omega)$ is positive. Therefore, the phase of $G(j\omega)$ is in magnitude less than $\pi/2$ radians for any finite $\omega \in \mathbb{R}$, as otherwise the locus would enter the other quadrants. For $D = 0$, the asymptotic phase as $\omega \to \infty$ is $\pi/2$ (the asymptote of $G_1(\infty) = 0$ is the imaginary axis). As a result, the relative degree (difference between the order of the polynomial in the denominator minus that of the numerator of $G(s)$) of such a transfer function when $D = 0$ is 1; otherwise, the asymptotic phase would be higher in magnitude than $\pi/2$, as an excess of at least two terms in the denominator would contribute to it, each with asymptotic phase $\pi/2$.

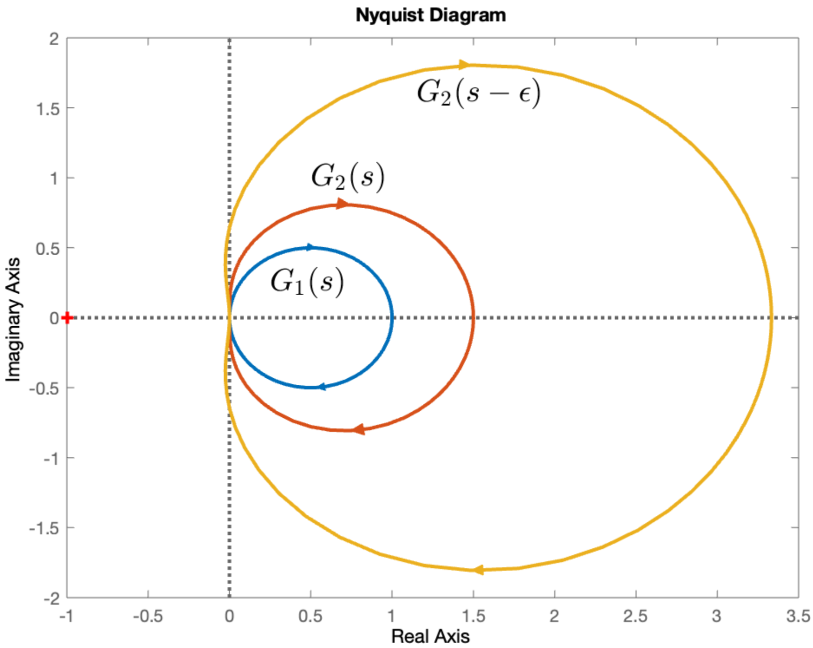

Figure 15: Blue: Nyquist diagram of the strictly positive real transfer function $G_1(s) = \frac{1}{s+1}$, which is of the form $G_1(s) = \frac{1}{s+a}$ with $a > 0$. Red: Nyquist diagram of the transfer function $G_2(s) = \frac{s+3}{(s+1)(s+2)}$, which is of the form $G_2(s) = \frac{s+a+b}{(s+a)(s+b)}$ with $a, b > 0$. This transfer function is not strictly positive real as it violates the last condition in the definition of strictly positive real transfer functions. As a result, for any $\epsilon > 0$ (here $\epsilon = 0.5$), the Nyquist diagram of $G_2(s-\epsilon)$ (yellow) enters the left-half plane.

conditions that involve the system's state space matrices.

Kalman-Yakubovich-Popov (KYP) Lemma. Consider a single-input, single-output LTI system with transfer function $G(s) = C(sI - A)^{-1}B + D$ with $D = 0$. Assume that (A, B) is controllable, and (A, C) is observable. We then have that $G(s)$ is strictly positive real if and only if there exist a matrix $P \in \mathbb{R}^{n\times n}$ with $P = P^\top \succ 0$, a vector $q \in \mathbb{R}^n$, and a scalar $\epsilon > 0$, such that

$$A^\top P + PA = -qq^\top - \epsilon P,$$
$$PB = C^\top.$$

Positive realness is related to passivity. $G(s)$ being positive real implies that the real part of $G(j\omega)$ has to be non-negative for all finite frequencies, and as such it can be realized as the impedance/conductance of an electric circuit with passive elements. In case of strict positive realness, $G(s)$ is realized by a strictly passive (dissipative) circuit, hence resistors are included. To see this, consider the transfer function $G_1(s)$ in Example 9 with $a = 1$, and notice that it can be thought of as

$G_1(s) = Y(s)/U(s)$ (with $U(s)$, $Y(s)$ being the Laplace transforms of the input and output signals, respectively) in the circuit of Example 8 in case there is no capacitor and $R = L = 1$. Such a system is strictly passive due to the presence of the resistor. It turns out that, unlike $G_1(s)$, the transfer function $G_2(s)$ in Example 9 cannot be realized by a strictly passive circuit.

Considering a system's frequency response, strict positive realness implies that for any finite ω the real part of $G(j\omega)$ is positive, hence its phase satisfies $|\angle G(j\omega)| < \pi/2$ (see also footnote after Example 9). To get some insights on why this is related to energy dissipation and thus strict passivity, take as example a case where $\angle G(j\omega) = 0$. This can be realized by an electric circuit with a sinusoidal voltage applied across a resistor (strictly passive system), as the output current is proportional to input voltage by Ohm's law and hence there will be no phase difference between the input and the output. The average power across the resistor would thus be non-zero (energy dissipation). On the other hand, if $|\angle G(j\omega)| = \pi/2$, we could replace the resistor with an inductor/capacitor (non-strictly passive system), but the average power would now be zero (no energy dissipation). The following result formalizes the relationship between strict positive realness and strict passivity.

Strictly Positive Realness $\implies$ Strict Passivity. Consider a single-input, single-output LTI system with matrices (A, B, C, D) and $D = 0$, that is the minimal realization of the transfer function $G(s) = C(sI - A)^{-1}B$.

- If $G(s)$ is strictly positive real, then the LTI system is strictly passive.

To see this, suppose $G(s)$ is strictly positive real. Let $P = P^\top \succ 0$, and consider a candidate $V(x) = \frac{1}{2}x^\top Px$ (quadratic as we have an LTI system). Notice that

$$\begin{aligned}
\dot{V}(x) &= \frac{1}{2}\dot{x}^\top Px + \frac{1}{2}x^\top P\dot{x} = \frac{1}{2}(Ax + Bu)^\top Px + \frac{1}{2}x^\top P(Ax + Bu) \\
&= x^\top PBu + \frac{1}{2}x^\top(A^\top P + PA)x \\
&= x^\top C^\top u + \frac{1}{2}x^\top(A^\top P + PA)x \\
&= u^\top y - \frac{1}{2}x^\top qq^\top x - \frac{1}{2}\epsilon x^\top Px \le u^\top y - \frac{1}{2}\epsilon x^\top Px,
\end{aligned}$$

where the second equality is since $\dot{x} = Ax + Bu$, and the third one is due to expanding the product and noticing that $x^\top PBu = u^\top B^\top Px$ since $u^\top B^\top Px$ is equal to its transpose as it is a scalar quantity. The fourth equality is using the fact that $PB = C^\top$ due to KYP's lemma, whose conditions are satisfied as $G(s)$ is assumed to be strictly positive real. The last equality is since $u^\top y = y^\top u = x^\top C^\top u$, and due to the first condition of KYP's lemma, while the inequality is since $qq^\top \succeq 0$. Setting $W(x) = \frac{1}{2}\epsilon x^\top P x$ and noticing that it is positive definite, we have that

$$u^\top y - W(x) \geq \dot{V}(x), \text{ for all } (x, u) \in \mathbb{R}^n \times \mathbb{R}^m,$$

thus establishing that the underlying LTI system is strictly passive (by the definition of strict passivity).

For a single-input, single-output LTI system with transfer function $G(s)$, having (A, B) controllable and (A, C) observable, is equivalent to (A, B, C, D) being a minimal realization (smallest order system that has the same transfer function) of $G(s)$ as there are no pole-zero cancellations. Hence, the controllability/observability assumption in KYP's lemma is consistent with the minimal realization statement in the result establishing a relationship between positive realness and strict passivity.

$$\begin{array}{ccc} \boxed{\text{LTI system strictly passive}} & \Longleftarrow & \boxed{G(s) \text{ strictly positive real}} \\ & & \Updownarrow \\ & & \boxed{\text{KYP's Lemma conditions}} \end{array}$$

Combining KYP's lemma with the result that establishes a relationship between positive realness and strict passivity, leads to the equivalences/implications in the diagram above. This suggests that for single-input, single-output LTI systems, the conditions of KYP's lemma offer the means to decide whether the underlying system is strictly passive. Determining whether a given (linear) system is passive will be exploited in the next section to analyze in terms of stability the feedback interconnection of an LTI system and a nonlinear controller of specific form. Therefore, KYP's lemma can be thought of as a generalization of Lyapunov's matrix equation that was used for stability analysis of autonomous LTI systems.

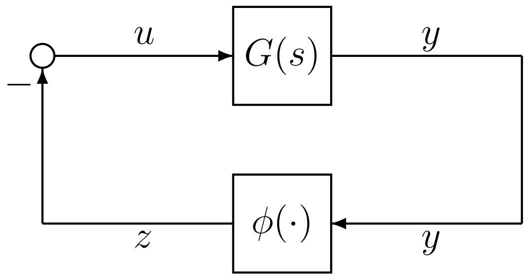

Figure 16: Feedback architecture of a system with open loop transfer function $G(s)$ and nonlinear feedback of the form $z = \phi(y)$.

4.3 Stability of linear systems with nonlinear feedback

4.3.1 Stability objectives and system reformulation

In this section we analyze in terms of stability the feedback interconnection of a single-input, single-output LTI system with a nonlinearity of specific form. With reference to Figure 16, consider:

Open loop system:

$$\dot{x} = Ax + Bu,$$
$$y = Cx.$$

Nonlinear feedback:

$$u = -z = -\phi(y).$$

Assume that (A, B, C, D), with $D = 0$ here, constitutes a minimal realization of a transfer function $G(s)$. The nonlinear feedback function $\phi(\cdot) : \mathbb{R} \to \mathbb{R}$ (notice that this is a scalar function as we are considering single-input, single-output systems) is termed memoryless or static as it does not contain any dynamics (it could, however, be time-varying). Notice the slight abuse of notation in Figure 16, where the system is encoded by a transfer function, while ϕ encodes a time-domain relationship. We consider functions ϕ that exhibit the following property.

Sector Bounded Nonlinearity. A memoryless function $\phi(\cdot) : \mathbb{R} \to \mathbb{R}$ belongs to the sector $[a, b]$ with $a < b$, if $\phi(0) = 0$, and

$$a \leq \frac{\phi(y)}{y} \leq b, \ \text{for all } y \neq 0.$$

Effectively, if a nonlinearity belongs to a sector, then this implies that $\phi(y)$ is bounded by two linear functions of y, with slopes a and b, respectively. Figure 17 provides a pictorial illustration of this fact.

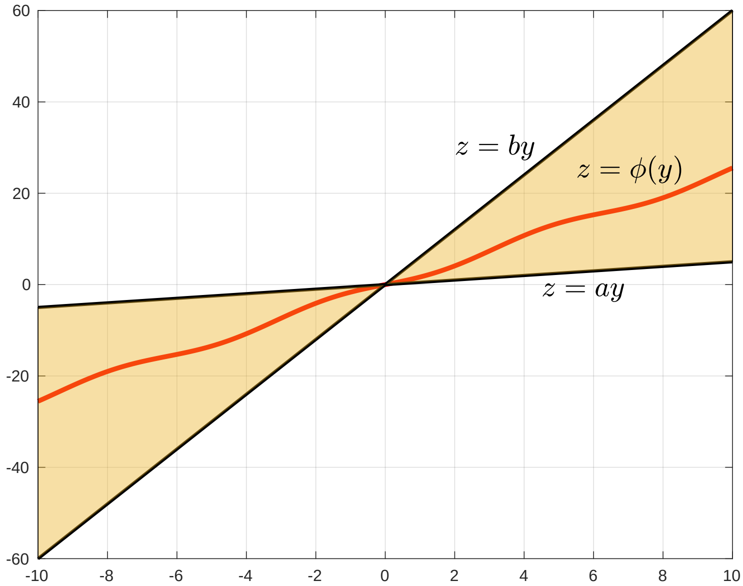

Figure 17: Pictorial illustration of a sector bounded nonlinear function ϕ.

The closed loop system is thus nonlinear, despite the fact that open loop system is an LTI one. Since ϕ is assumed to be a sector bounded nolinearity, and hence $\phi(0) = 0$, the origin $x^\star = 0$ is also an equilibrium of the closed loop system. We aim at analyzing the stability properties of the origin for systems of this form, and in fact to determine conditions under which this is an asymptotically stable equilibrium for the closed loop system. In particular, we will perform this analysis not for a particular nonlinear function ϕ, but for any nonlinear function within a given sector $[a, b]$, with $b > 0$. This problem is often called Lure's problem.

Several stability criteria have been determined for this problem. We will derive in the next subsection one of these, known as the circle criterion. This criterion has a graphical interpretation that generalizes the Nyquist criterion for stability of linear feedback systems to linear systems with sector bounded feedback nonlinearities. The circle criterion is based on the concept of passivity introduced in the previous chapter. In particular, if the transfer function $G(s)$ is strictly positive real (recall that strict positive realness implies strict passivity), and the nonlinearity ϕ belongs to the sector $[0, \infty)$, then the origin is an asymptotically stable equilibrium of the closed-loop system. To see this, notice that

- Strictly positive realness is equivalent to the conditions in KYP's lemma being satisfied (see discussion in previous chapter).

- Since we have an LTI system, and due to the strong connection between storage and Lyapunov functions, consider a quadratic candidate storage function $V(x) = \frac{1}{2}x^\top Px$, where $P = P^\top \succ 0$.
- Under this choice of a storage function, the conditions of the KYP's lemma lead to the derivations in p. 61, which resulted in
$$\dot{V}(x) \leq u^\top y - \frac{1}{2}\epsilon x^\top Px.$$
- If the nonlinearity belongs to the sector $[0, \infty)$, then it follows that $y\phi(y) \geq 0$ for any y. For the closed loop system we substitute $u = -\phi(y)$ in the previous inequality (recall that y and $\phi(y)$ are scalars), obtaining
$$\dot{V}(x) \leq -y\phi(y) - \frac{1}{2}\epsilon x^\top Px < 0,$$
where the last inequality is strict since $\epsilon > 0$ and $P \succ 0$. By Lyapunov's direct method in Theorem 3, we then obtain that the origin is an asymptotically stable equilibrium point of the closed loop system.

Here we aim at extending this line of arguments to the case where ϕ belongs to a generic sector $[a, b]$ and not necessarily to $[0, \infty)$, while the open loop transfer function $G(s)$ is not necessarily strictly positive real, allowing it to be even unstable (having poles with positive real part). To this end, we construct an equivalent closed-loop system with a new open loop transfer function $\widehat{G}(s)$, and a new feedback nonlinearity $\widehat{\phi}$ belonging to the sector $[0, \infty)$. This is achieved by adding feedforward and feedback loops to the subsystems of Figure 16, obtaining the loop transformation of Figure 18.

The reformulated system is equivalent to the original one in the sense that, given the same initial conditions, the time domain signals u, y and z in the loop reformulation, would be identical to those of the original system. To see this, following the junctions in Figure 18,

$$u = -ay - (z - ay) = -z = -\phi(y),$$

where the term in brackets is the signal that enters the negative input of the left-most junction. Therefore, u and y are linked with exactly the same relationship as with the system of Figure 16, and hence analyzing the closed loop system of the loop transformation in terms of stability will immediately imply the same stability conclusions for the original closed system that we are interested in.

To see that the reformulated system has a nonlinearity $\widehat{\phi}$ belonging to the sector $[0,\infty)$, consider Figure 18. Denote by $\widehat{z}$, $\widehat{y}$ the input and output, respectively, of the outer feedback loop (yellow) in Figure 18, and notice that $\widehat{z}$ coincides with the output of the inner feedback loop (gray). By the left junction in the feedback block this is in turn equal to $\phi(y) - ay$, and since $\phi(y) \in [ay, by]$, we would have that $\widehat{z} = \phi(y) - ay \in [0, (b-a)y]$. We wish to establish though that $\widehat{z} = \widehat{\phi}(\widehat{y}) \in [0,\infty)$. We have the following implications

$$\widehat{z} \in [0, (b-a)y] \iff y \in \Big[\frac{1}{b-a}\widehat{z}, \infty\Big),$$
$$\widehat{z} = \widehat{\phi}(\widehat{y}) \iff \widehat{y} = \widehat{\phi}^{-1}(\widehat{z}) = y - \frac{1}{b-a}\widehat{z},$$

where the last equality is due to the right-most junction in the $\widehat{\phi}$ block. By $\widehat{\phi}^{-1}$ we mean the inverse function of $\widehat{\phi}$, which we assume is well defined. Combining these, we obtain that $\widehat{y}$ belongs to the sector $[0,\infty)$. Since $\widehat{y} = \widehat{\phi}^{-1}(\widehat{z})$, this is in turn equivalent to $\widehat{\phi}(\widehat{y}) = \widehat{z}$ belonging also to the $[0,\infty)$, thus establishing our claim.

4.3.2 Stability assessment

Consider the loop transformation in Figure 18. We will investigate the stability properties of the closed loop system of this loop transformation, as the same stability conclusions apply for the original closed loop system. The reformulated architecture involves the feedback interconnection of a new open loop system with transfer function $\widehat{G}(s)$, and a nonlinear feedback function $\widehat{\phi}$, which is now in the sector $[0,\infty)$. As argued in the previous subsection, the origin is an asymptotically stable equilibrium point of the closed loop system, provided that $\widehat{G}(s)$ is a strictly positive real transfer function. To this end, recall that $a \neq b$, and notice that

$$\widehat{G}(s) = \frac{G(s)}{1+aG(s)} + \frac{1}{b-a} = \frac{1}{b-a}\frac{1+bG(s)}{1+aG(s)},$$

where the first term in the first equality is the negative feedback interconnection of the original transfer function and a feedback gain a, while the second term is

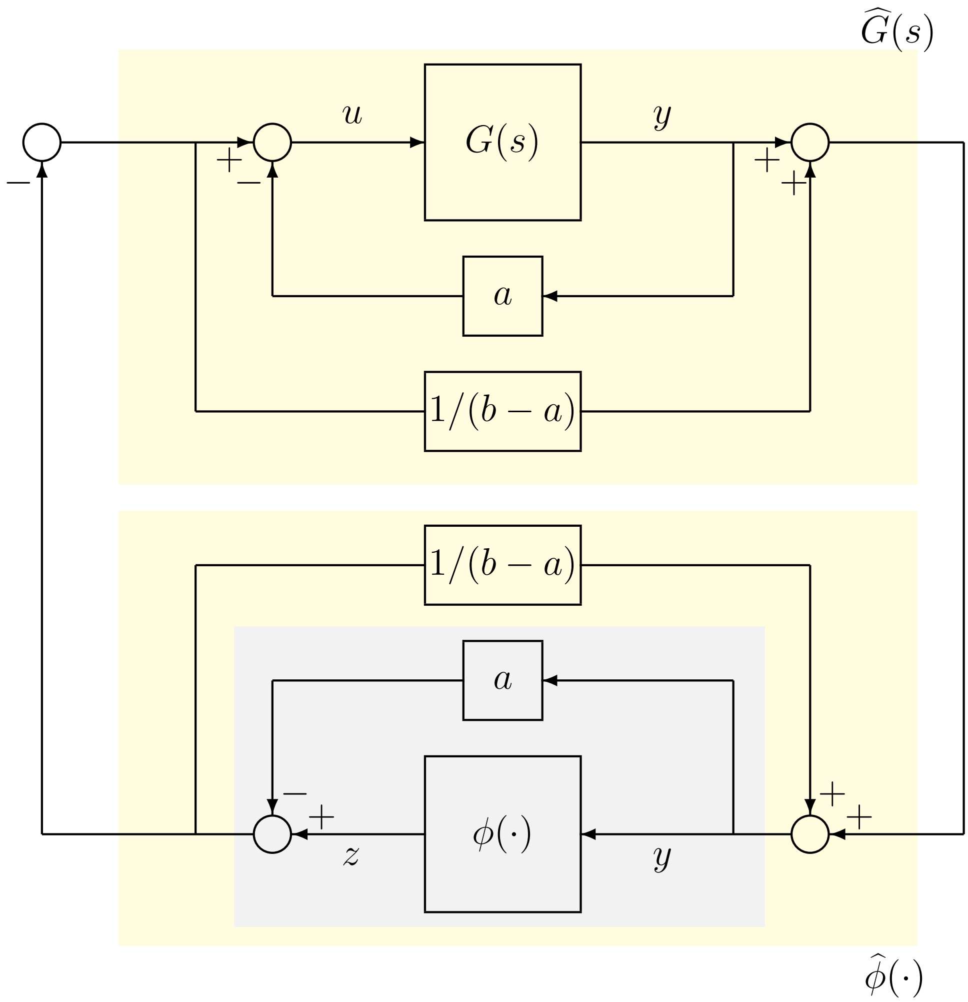

Figure 18: Loop transformation of the system in Figure 16. If ϕ belongs to the sector $[a, b]$ then $\widehat{\phi}$ belongs to the sector $[0, \infty)$. The new system has open loop transfer function $\widehat{G}(s)$ and nonlinear feedback encoded by the function $\widehat{\phi}$.

the feedforward constant $\frac{1}{b-a}$. Notice also that since we have assumed that $D = 0$, the original transfer function $G(s)$ would be strictly proper, i.e., the degree of the polynomial in the numerator is smaller than that of the denominator. As a result $G(\infty) = 0$. However, the reformulated transfer function $\widehat{G}(s)$ will no longer be strictly proper due to the presence of the feedforward term. Since $G(\infty) = 0$, this implies that $\widehat{G}(\infty) = \frac{1}{b-a} > 0$ as we have assumed that $b > a$.

By the definition of strict positive realness $\widehat{G}(s)$ is required to satisfy certain conditions. We will check these separately.

1. $\widehat{G}(\infty) > 0$. This limiting case is directly satisfied as justified above.
2. All poles of $\widehat{G}(s)$ have negative real part. Note that $\widehat{G}(s)$ is the negative feedback

interconnection of $G(s)$ and the gain a. If $a \neq 0$, then according to Nyquist's stability criterion, all poles of $\widehat{G}(s)$ have negative real part if and only if the Nyquist plot of (the open loop transfer function) $G(s)$ encircles (without passing through) the $-1/a$ point as ω goes from $-\infty$ to $+\infty$, μ times anti-clockwise[§], where $\mu \geq 0$ denotes the number of poles of $G(s)$ that have positive real part. Therefore, the poles of $\widehat{G}(s)$ could all have negative real part, even if this is not the case for $G(s)$. If $a = 0$, the poles of $\widehat{G}(s)$ coincide with these of $G(s)$, hence in that case all poles of $G(s)$ need to have negative real part for this to be the case for $\widehat{G}(s)$ as well.

3. $\mathrm{Re}[\widehat{G}(j\omega)] > 0$ for all $\omega \in \mathbb{R}$. We will investigate this by considering different cases according to the values of a and b, as performed below.

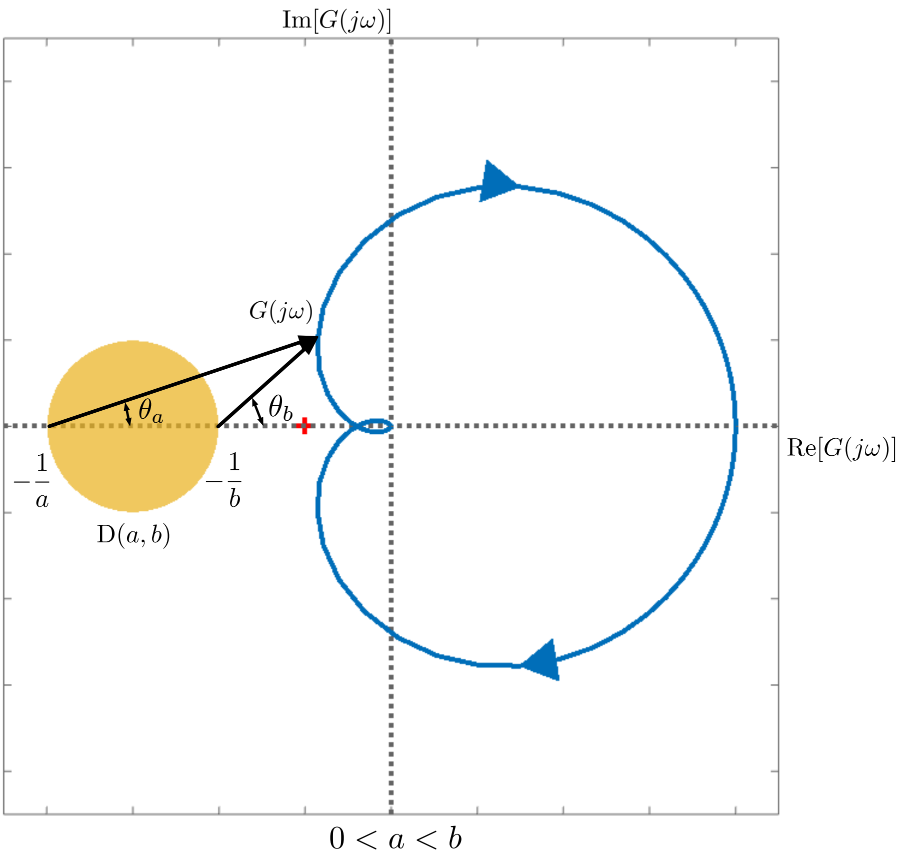

Figure 19: Pictorial illustration of the fact that for the case where $0 < a < b$, to have that $\mathrm{Re}[\widehat{G}(j\omega)] > 0$ for all $\omega \in \mathbb{R}$, the Nyquist plot of $G(s)$ should remain outside the closed disk $\mathrm{D}(a, b)$.

[§]Note that if $G(s)$ has poles on the imaginary axis then the Nyquist plot is assumed to "close" through the right-half plane as discussed in the footnote in p. 57. Moreover, encirclements are counted algebraically; as an example one clockwise and one anti-clockwise encirclement, imply a net number of zero anti-clockwise encirclements. Moreover, we refer to $G(s)$ encircling the $-1/a$ point, as this then is equivalent to $aG(s)$ encircling the -1 point.

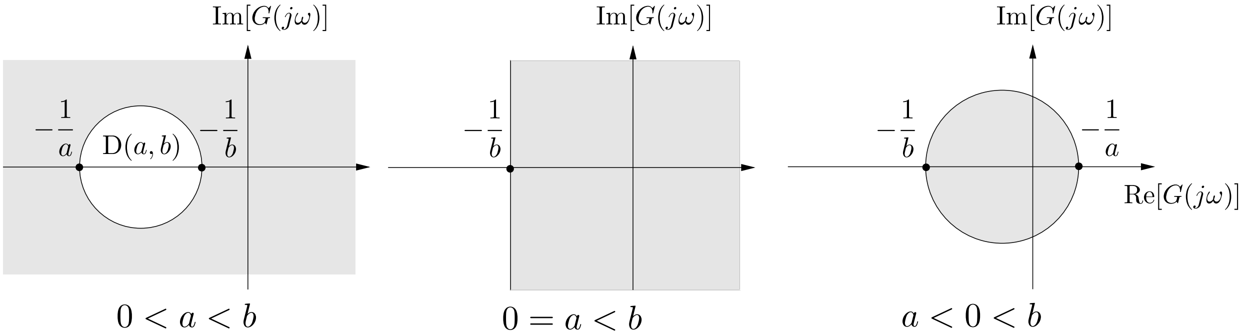

Figure 20: Pictorial illustration of the cases in the circle criterion. In each case, the Nyquist plot of $G(s)$ should be contained in the corresponding shaded part. Note that with the notation $\mathrm{D}(a,b)$ we imply a disk parameterized by a, b, as shown in the figure. The dependency of its diameter's extrema on a, b would depend on the sign of these parameters (see left and right panels).

Case $0 < a < b$: Notice that by pulling a and b as common factors in the denominator and numerator of $\widehat{G}(s)$, respectively, $\mathrm{Re}[\widehat{G}(j\omega)] > 0$ is equivalent to

$$\frac{b}{a}\mathrm{Re}\Big[\frac{\frac{1}{b}+G(j\omega)}{\frac{1}{a}+G(j\omega)}\Big] > 0 \iff \mathrm{Re}\Big[\frac{\frac{1}{b}+G(j\omega)}{\frac{1}{a}+G(j\omega)}\Big] > 0,$$

where the implication is since $\frac{b}{a} > 0$. For this condition to be satisfied, the Nyquist plot of $G(s)$ should not enter the closed disk $\mathrm{D}(a,b)$ shown in Figure 19. This admits a geometric justification; to this end, refer to Figure 19. The numerator and the denominator of $\widehat{G}(j\omega)$ are the complex vectors indicated by the figure arrows. To see this, notice that the denominator (similarly for the numerator), can be thought of as the vector sum of a vector emanating from the point $(-\frac{1}{b}, 0)$ to the origin, and another one from the origin to the point $G(j\omega)$ on the Nyquist plot. Let $\theta_b = \angle(\frac{1}{b}+G(j\omega))$ and $\theta_a = \angle(\frac{1}{a}+G(j\omega))$ denote the arguments of the numerator and the denominator of $\widehat{G}(j\omega)$, respectively. The condition $\mathrm{Re}[\widehat{G}(j\omega)] > 0$ for all $\omega \in \mathbb{R}$ is equivalent to $\theta_b - \theta_a < \pi/2$, as otherwise the cosine of the angle between the numerator and the denominator, and hence the sign of the real part of $\widehat{G}(j\omega)$, would become non-positive. This is in turn satisfied if the Nyquist plot of $G(s)$ lies outside the closed disk $\mathrm{D}(a,b)$, as if there is a point on the perimeter or inside the disk, then the angle difference becomes at least $\pi/2$. This is also illustrated by the shaded region in the left panel of Figure 20.

Case $0 = a < b$: In this case, since $a = 0$, we have that $\widehat{G}(s) = \frac{1}{b} + G(s)$. Therefore, $\mathrm{Re}[\widehat{G}(j\omega)] > 0$ is equivalent to

$$\mathrm{Re}\Big[\frac{1}{b} + G(j\omega)\Big] > 0 \iff \mathrm{Re}[G(j\omega)] > -\frac{1}{b}.$$

Therefore, the Nyquist plot should be contained in the open halfspace illustrated by the shaded region in the middle panel of Figure 20.

Case $\mathbf{a < 0 < b}$: Proceeding as in the case $0 < a < b$, $\mathrm{Re}[\widehat{G}(j\omega)] > 0$ is now equivalent to

$$\frac{b}{a}\mathrm{Re}\Big[\frac{\frac{1}{b} + G(j\omega)}{\frac{1}{a} + G(j\omega)}\Big] > 0 \iff \mathrm{Re}\Big[\frac{\frac{1}{b} + G(j\omega)}{\frac{1}{a} + G(j\omega)}\Big] < 0,$$

where the inequality is reversed compared to the case $0 < a < b$, as we now have $\frac{b}{a} < 0$. Following the same arguments with the case $0 < a < b$, but with the inequality now reversed, we can conclude that the Nyquist plot must lie entirely in the interior (as the inequality is strict) of the disk $\mathrm{D}(a, b)$. This is illustrated by the shaded region in the right pabel of Figure 20. Consequently, the Nyquist plot cannot encircle the $-1/a$ point; for this to be consistent with the condition that needs to be satisfied for the poles of $\widehat{G}(s)$ to have negative real part, we must have that the open loop system $G(s)$ must have all its poles with negative real part as well. Notice that this is not necessarily the case when $0 < a < b$.

These observations are collected in the following theorem.

Theorem 9 (Circle Criterion)**.** *Consider the closed loop system in Figure 16 and its loop transformation in Figure 18. Let $x^\star = 0$ be an equilibrium point of the closed loop system, and further assume that the nonlinearity ϕ belongs to the sector $[a, b]$.*
If one of the following is satisfied as appropriate depending on a and b:

- *if $0 < a < b$, the Nyquist plot of $G(s)$ does not enter the disc $\mathrm{D}(a, b)$ (Figure 20, left panel) and encircles it μ times anti-clockwise, where $\mu \geq 0$ is the number of poles of $G(s)$ with positive real part;*
- *if $0 = a < b$, all poles of $G(s)$ have negative real part, and the Nyquist plot of $G(s)$ is contained in the halfspace $\mathrm{Re}[G(j\omega)] > -\frac{1}{b}$ (Figure 20, middle panel);*
- *if $a < 0 < b$, all poles of $G(s)$ have negative real part, and the Nyquist plot of $G(s)$ stays in the interior of the disc $\mathrm{D}(a, b)$ (Figure 20, right panel),*

then the equilibrium point $x^\star = 0$ of the closed-loop system is (globally) asymptotically stable.

Note that if $a < b < 0$, then we can still invoke the same theorem but for the system $-G(s)$ instead. In particular, $x^\star = 0$ would be asymptotically stable for the closed loop system, if the Nyquist plot of $-G(s)$ does not enter $\mathrm{D}(-a, -b)$ and encircles it μ times anti-clockwise, where $\mu \geq 0$ is the number of poles of $G(s)$ with positive real part. The reason for the negative signs is that now the feedback gain a in the loop transformation of Figure 18 becomes negative.

As b tends to a, the feedback nonlinearity in the original system tends to become linear (as the two slopes of the sector bound tend to each other). The disk gradually then degenerates to the point $-1/a$, and the circle criterion reduces to the Nyquist one as we now tend to a linear closed loop system.

Example 10. *Consider a feedback system in the form of Figure 16, with*

$$G(s) = \frac{24}{(s-1)(s+2)(s+3)},$$

and with ϕ belonging to the sector $[0.275, 0.3]$. Determine whether the origin is an asymptotically stable equilibrium of the closed loop system.

Solution: *To determine the stability properties of the closed loop system we apply the circle criterion in Theorem 9. We first sketch the Nyquist diagram of the open loop transfer function $G(s)$. We do this via the MATLAB commands*

$$\mathrm{G = zpk(tf([24], [1\ \ 4\ \ 1\ \ \ -6]));}$$
$$\mathrm{nyquist(G);}$$

and obtain the plot shown in Figure 21. Notice that in this case $a = 0.275$ and $b = 0.3$, so we need to check the condition of the theorem corresponding to the case where $0 < a < b$.
The open loop system has $\mu = 1$ poles with positive real part, namely, $s = 1$. As such, we require the Nyquist diagram to encircle the $-\frac{1}{a}$ point one time counterclockwise. Moreover we would like the Nyquist plot to remain outside the disk $\mathrm{D}(a, b)$ (shown with yellow in Figure 21). By inspection of Figure 21, both these requirements are met, hence by means of Theorem 9 we infer that the origin is an asymptotically stable equilibrium of the closed loop system.

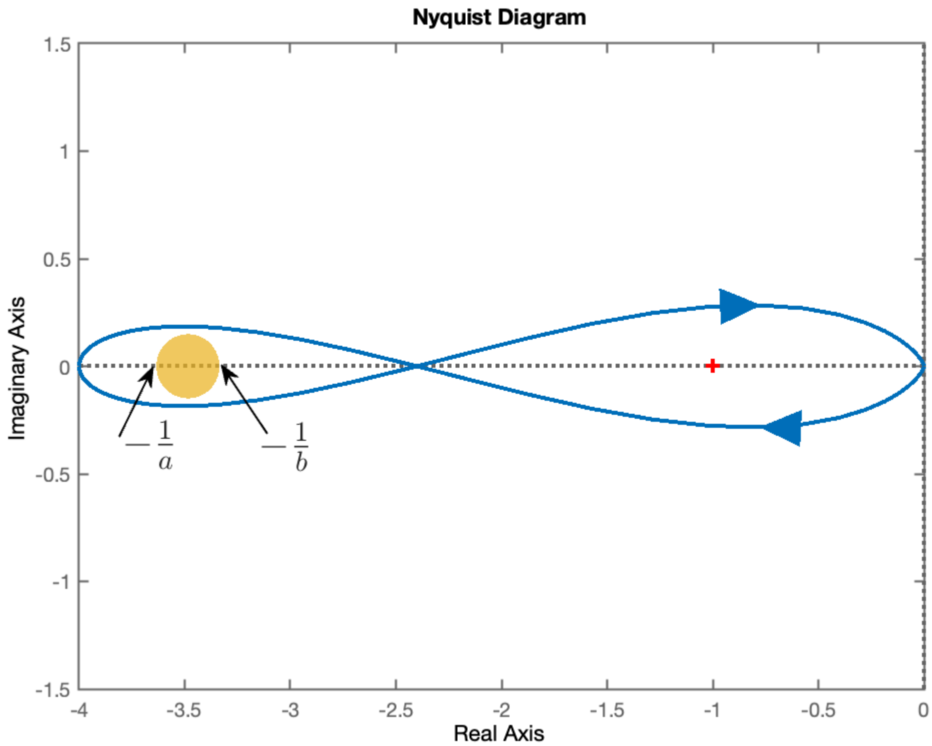

Figure 21: Nyquist diagram of the transfer function $G(s) = \frac{24}{(s-1)(s+2)(s+3)}$ in Example 10; sector bounded nonlinearity with $a = 0.275$ and $b = 0.3$, giving rise to the "yellow" disk.

4.4 Summary

In this chapter we first considered autonomous, linear time invariant systems of the form $\dot{x}(t) = Ax(t)$. We showed that that stability in this case can be analyzed by considering a quadratic Lyapunov function of the form $V(x) = x^\top Px$, where $P = P^\top \succ 0$. We established the following stability criterion.

Lyapunov Stability of (Autonomous) LTI Systems (Theorem 4)
Consider an LTI system $\dot{x}(t) = Ax(t)$. The system is asymptotically stable if and only if for any given $Q = Q^\top \succ 0$, there exists a unique $P = P^\top \succ 0$ satisfying the Lyapunov matrix equation

$$A^\top P + PA = -Q.$$

We then analyzed systems with inputs and established the notion of passivity. In particular, a (possibly nonlinear) system was termed strictly passive if there exists a continuously differentiable function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$, $V(x) \geq 0$

for all $x \in \mathbb{R}^n$, as well as a positive definite function $W(\cdot) : \mathbb{R}^n \to \mathbb{R}$ such that

$$u^\top y - W(x) \geq \dot{V}(x) = \nabla V(x) f(x,u), \text{ for all } (x,u) \in \mathbb{R}^n \times \mathbb{R}^m.$$

For linear systems systems, with a transfer function $G(s)$, we established that

$$\text{strict positive realness of } G(s) \quad \Longrightarrow \quad \text{strict passivity.}$$

We provided conditions for a transfer function to be strictly positive real, and showed that by means of Kalman-Yakubovich-Popov's lemma, there exist equivalent time-domain algebraic conditions.

We then considered feedback systems with a linear open-loop system, and a memoryless feedback nonlinearity. We showed that if the underlying system enjoys certain passivity properties, and the nonlinearity is sector bounded, then this allows us to draw certain stability conclusions. For systems of this type we presented the circle criterion to decide about stability graphically.

Circle Criterion (Theorem 9)
Consider the closed loop system in Figure 16 and its loop transformation in Figure 18. Let $x^\star = 0$ be an equilibrium point of the closed loop system, and further assume that the nonlinearity ϕ belongs to the sector $[a,b]$.
If one of the following is satisfied as appropriate depending on a and b:

- if $0 < a < b$, the Nyquist plot of $G(s)$ does not enter the disc $\mathrm{D}(a,b)$ (Figure 20, left panel) and encircles it μ times anti-clockwise, where $\mu \geq 0$ is the number of poles of $G(s)$ with positive real part;
- if $0 = a < b$, all poles of $G(s)$ have negative real part, and the Nyquist plot of $G(s)$ is contained in the halfspace $\mathrm{Re}[G(j\omega)] > -\frac{1}{b}$ (Figure 20, middle panel);
- if $a < 0 < b$, all poles of $G(s)$ have negative real part, and the Nyquist plot of $G(s)$ stays in the interior of the disc $\mathrm{D}(a,b)$ (Figure 20, right panel),

then the equilibrium point $x^\star = 0$ of the closed-loop system is (globally) asymptotically stable.

5 Appendix

5.1 Selected results on continuity of functions

We provide some continuity definitions (and some of their implications) as a reference for some of the terms used at different parts in these notes. In these definitions, we use the set $S \subseteq \mathbb{R}^k$ (where k is generic) to denote the domain of the functions involved. Recall that a set $S \subseteq \mathbb{R}^k$ is open if for any point $x \in S$, there exists a (possibly small) neighbourhood around that point so that the entire neighbourhood is contained in S. As example, consider the intervals $(-1, 1)$, or $(-\infty, 1)$ on the real line. It is closed if its set complement is open. As example, consider the sets $[-1, 1]$, $[1, +\infty)$. There are sets that are neither open, nor closed, e.g., $(-1, 1]$. If a set is closed *and* bounded, i.e., does not extend to infinity, then it is called compact. For example consider the set $[-1, 1]$.

In the following, the domain S is not restricted in any of the aforementioned classes unless stated otherwise. Moreover, whenever a norm $\|\cdot\|$ is used, this is to be understood as the Euclidean norm.

Continuity. A function $f(\cdot) : S \to \mathbb{R}^m$ is said to be continuous on S if for all $\hat{x} \in S$, for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $x \in S$,

$$\|x - \hat{x}\| < \delta \implies \|f(x) - f(\hat{x})\| < \epsilon.$$

Uniform continuity. A function $f(\cdot) : S \to \mathbb{R}^m$ is said to be uniformly continuous on S if for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $x, \hat{x} \in S$,

$$\|x - \hat{x}\| < \delta \implies \|f(x) - f(\hat{x})\| < \epsilon.$$

Lipschitz continuity. A function $f(\cdot) : S \to \mathbb{R}^m$ is said to be Lipschitz continuous on S if there exists a (finite) scalar $L > 0$ such that for all $x, \hat{x} \in S$,

$$\|f(x) - f(\hat{x})\| \leq L\|x - \hat{x}\|.$$

L is then called the Lipschitz constant of f.

Intuitively, continuity implies that we can always pick $\hat{x}$ close enough to a given

x (δ-close), if we would like $f(x)$ and $f(\hat{x})$ to remain close (ϵ-close). Uniform continuity is stronger than continuity. Namely, the choice of δ is independent from that of $\hat{x}$, requiring that the same δ should be chosen for all $\hat{x} \in S$. To see this, notice that the order of the clauses "for all $\hat{x} \in S$" and "there exists $\delta > 0$", is reversed in the two definitions. As such, not all continuous functions are uniformly continuous; see Example 12.

Lipschitz continuity implies that a function not only is continuous, but also exhibits a certain (linear) growth property so that it does not become "too steep". According to this definition the same Lipschitz constant exists for all points $x, \hat{x} \in S$. If $S = \mathbb{R}^k$ in the Lipschitz continuity definition, then we often refer to the associated function as globally Lipschitz continuous. If a function is Lipschitz continuous, then it is also uniformly continuous (and hence continuous). See Example 11 for a proof of this argument; the opposite, however is not true in general. If a function is differentiable and also has bounded derivatives, then it is also Lipschitz continuous. In general, the following implications summarize the relationships among the different notions of continuity introduced above (over the same domain):

$$\begin{array}{ccc} \text{Continuity} & \Longleftarrow & \text{Uniform continuity} \\ & & \Uparrow \\ \text{Differentiability + bounded derivatives} & \Longrightarrow & \text{Lipschitz continuity} \end{array}$$

Example 11. *Show that if a function f is Lipschitz continuous on a domain $S \subseteq \mathbb{R}^n$, then it is also uniformly continuous over the same domain.*

Solution: *Consider any f that is Lipschitz continuous on some domain S, with constant L. Notice that the uniform continuity definition requires that δ depends only on ϵ (for each ϵ a possibly different δ may exist), and is independent of $x, \hat{x}$. Fix any $\epsilon > 0$, and choose $\delta = \frac{\epsilon}{L}$. For any $\|x - \hat{x}\| < \delta$,*

$$\|f(x) - f(\hat{x})\| \leq L\|x - \hat{x}\| < L\delta = \epsilon,$$

where the equality is since $\delta = \frac{\epsilon}{L}$. This then implies that f is uniformly continuous on S.

✎ **Example 12.** *Show that the function $f(x) = x^2$ is **not** uniformly continuous on $S = \mathbb{R}$.*

Solution: *The function $f(x) = x^2$ is continuous on $\mathbb{R}$ (this can also be verified either by definition or graphically). Here we will show that it is not uniformly continuous on $\mathbb{R}$. This is equivalent to showing that the "negation" of the uniform continuity definition's holds, namely, that there exists $\epsilon > 0$, such that for any $\delta > 0$, there exist points $x, \hat{x} \in \mathbb{R}$, such that*

$$\|x - \hat{x}\| < \delta \quad \text{and} \quad \|f(x) - f(\hat{x})\| \geq \epsilon,$$

Therefore, fix $\epsilon = 1$, and consider any $\delta > 0$. Let then $\hat{x} = \frac{1}{\delta}$ and $x = \hat{x} + \frac{\delta}{2}$. Under these choices we have $\|x - \hat{x}\| = \frac{\delta}{2} < \delta$, and

$$\|f(x) - f(\hat{x})\| = \|x^2 - \hat{x}^2\| = \left\| \left(\frac{1}{\delta} + \frac{\delta}{2}\right)^2 - \frac{1}{\delta^2} \right\| = 1 + \frac{\delta^2}{4} > 1 = \epsilon.$$

We have thus constructed an instance where the uniform continuity requirement does not hold; hence $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

The following two cases are of interest.

- **Continuity on compact sets:** Let S be a compact set. We then have that if f is continuous on S it is also uniformly continuous.
- **Continuity of composition of functions:** Let $g(\cdot) : S \to S_g$, $f(\cdot) : S_f \to \mathbb{R}^m$ be continuous functions on their domains, and assume $S_g \subseteq S_f$ so that the function composition $(f \circ g)(\cdot) : S \to \mathbb{R}^m$ is well defined (recall that $(f \circ g)(x) = f(g(x))$ for all $x \in S$). We then have that $f \circ g$ is also continuous on S, i.e., the composition of continuous functions preserves continuity. Similarly, if two functions are uniformly continuous on their respective domains, their composition will be uniformly continuous as well.

5.2 Existence, uniqueness & continuity properties of solutions to ODEs

We highlight how continuity is important when it comes to existence and uniqueness of solutions to ordinary differential equations, and also discuss the continuity properties of the resulting solutions. Consider a nonlinear dynamical system with state $x(t) \in \mathbb{R}^n$, $t \in \mathbb{R}$, satisfying

$$\begin{aligned} x(t_0) &= x_0, \\ \dot{x}(t) &= f(x(t), t), \text{ for all } t \in \mathbb{R}, \end{aligned}$$

where (t_0, x_0) is an arbitrary initial condition, and $f(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ denotes the system dynamics. Assume that f is continuous with respect to both arguments on $\mathbb{R}^n \times \mathbb{R}$, and in addition, for any fixed t, it is Lipschitz continuous with respect to the first argument on $\mathbb{R}^n$. We then have that for any initial condition there exists a unique continuous solution $x(\cdot) : \mathbb{R} \to \mathbb{R}^n$ that is compatible with the initial condition and satisfies the system dynamics.

Assume in addition that:

1. The state is contained in a bounded domain. To this end, there exists a finite scalar $r \in \mathbb{R}$ such that $\|x(t) - x^\star\| < r$, for all $t \geq 0$.
2. The function $f(\cdot, t)$ is Lipschitz continuous with respect to its first argument uniformly with respect to t, i.e., the Lipschitz constant is the same for all $t \geq 0$, over the set $\|x - x^\star\| < r$.

Under these additional assumptions, $x(t)$ is a uniformly continuous function of t over the bounded domain. To see this, denote by $x^\star$ a system equilibrium. Consider now the Lipschitz continuity definition with $\hat{x} = x^\star$. We have that for any $t \in \mathbb{R}$,

$$\|f(x(t), t)\| = \|f(x(t), t) - f(x^\star, t)\| \leq L\|x(t) - x^\star\| < Lr.$$

where the equality is since $f(x^\star, t) = 0$ for all $t \geq 0$, as $x^\star$ is an equilibrium, the inequality is since f is Lipschitz with respect to the first argument uniformly with respect to t (the Lipschitz constant L is independent of t), and the strict inequality is since $\|x(t) - x^\star\| < r$ for any t. This derivation implies that f (which is $\dot{x}(t)$) is in turn bounded. As such, the state solution $x(\cdot)$ is differentiable (by existence and uniqueness of solutions it satisfies $\dot{x}(t) = f(x(t), t)$), and its derivative f was shown to be bounded. By the continuity implications diagram, this in turn implies that the solution $x(t)$ is a uniformly continuous function of t.

5.3 Proof of Theorem 8

Let $x^\star = 0$ be an equilibrium of $\dot{x}(t) = f(x(t), 0)$. Assume that there exists an open set $S \subseteq \mathbb{R}^n$ that contains the equilibrium, i.e., $x^\star \in S$. Assume also that the system is strictly passive.This implies then there exists a storage function V and a positive definite function W (with domain S) that when $u = 0$, satisfy $\dot{V}(x) \leq -W(x)$ for all $x \in S$. Therefore, since $W(x) > 0$ for any $x \in S$ with $x \neq x^\star$, we would have that $\dot{V}(x) < 0$ for all $x \in S$ with $x \neq x^\star$.

As such, and since $V(0) = 0$, the storage function $V(x)$ would be a valid choice for a Lyapunov function if $V(x) > 0$ (notice that the inequality should be strict) for all $x \in S$ with $x \neq x^\star = 0$. We will show that under strict passivity this is indeed the case. By Lyapunov's direct method (Theorem 2) we would then establish that $x^\star = 0$ is an asymptotically stable equilibrium point.

We thus show that $V(x) > 0$ for all $x \in S$ with $x \neq 0$. Fix any $\bar{x} \in S$, and let $x(t)$ denote the (unique) solution of $\dot{x}(t) = f(x(t), 0)$ over $[0, T]$ (for some $T > 0$) starting from $x(0) = \bar{x}$. Integrating the inequality $\dot{V}(x) \leq -W(x)$ results in

$$V(x(t)) - V(\bar{x}) \leq -\int_0^t W(x(\tau))\, \mathrm{d}\tau, \text{ for all } t \in [0, T].$$

Since V is a storage function, $V(x(t)) \geq 0$. Therefore, we have that

$$V(\bar{x}) \geq \int_0^t W(x(\tau))\, \mathrm{d}\tau, \text{ for all } t \in [0, T].$$

Assume for the sake of contradiction that that there exists $\bar{x} \in S$ with $\bar{x} \neq 0$ such that $V(\bar{x}) = 0$. Since W is positive definite, the previous inequality implies then that $W(x(t)) = 0$ for all $t \in [0, T]$. This in turn implies that $x(t) = 0$ for all $t \in [0, T]$; for $t = 0$ this results in $\bar{x} = x(0) = 0$. The latter contradicts the hypothesis that $\bar{x} \neq 0$, and therefore we need to have that $V(x) > 0$ for all $x \in S$ with $x \neq 0$, thus concluding the proof. This establishes that under strict passivity, a storage function qualifies as a valid Lyapunov function.

5.4 On the need of both conditions in the asymptotic stability definition

The asymptotic stability definition in Chapter 2 not only requires that state trajectories tend to an equilibrium (condition 2), but that the equilibrium is also stable (condition 1). This may seem redundant, however, convergence to the equilibrium is *not* sufficient for equilibrium stability. To gain some insights on this, consider the second order system below with the origin being its equilibrium point.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_1^2(t) - x_2^2(t) \\ 2x_1(t)x_2(t) \end{bmatrix}.$$

The phase portrait of this system is as in Figure 22. Notice that from any initial state that is not on the real axis, state trajectories tend to the origin; some sample trajectories with solid lines in Figure 22. From initial states on the positive real axis, trajectories tend to $+\infty$, while from the negative real axis tend to the origin. Assuming that $-\infty$, $+\infty$ correspond to the same point, then all state trajectories, no matter where they start, tend to the origin possibly through the "point of infinity" (dashed lines in Figure 22). This shows that the second condition, namely convergence to the equilibrium, in the asymptotic stability definition is satisfied.

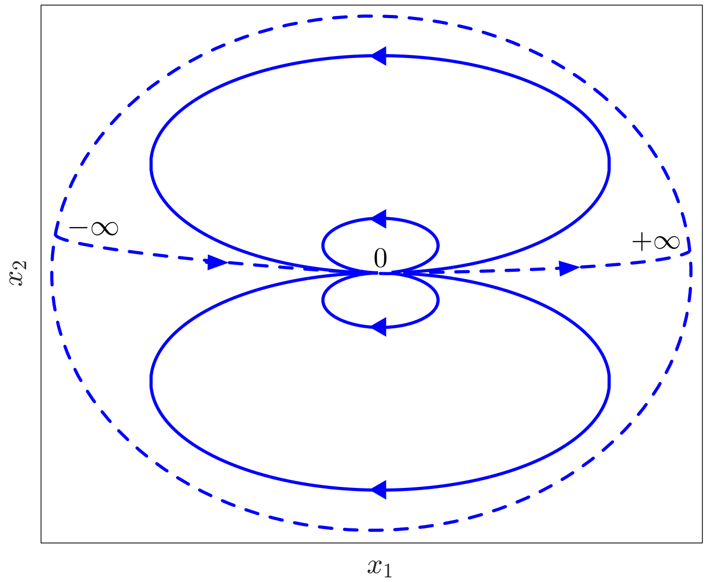

Figure 22: Example of an equilibrium point (the origin) that is not stable, however, all state trajectories still tend to it if we consider that $-\infty$ and $+\infty$ is the same point, namely, the "point of infinity".

However, the origin is not a stable equilibrium. To see this, notice that for any $\epsilon > 0$, no matter how small δ is chosen, there always exist some initial state within this δ-ball and close to the positive real axis, so that the resulting trajectory will escape the ϵ-ball prior to converging to the origin (potentially through the "point of infinity"). This demonstrates the need for both conditions in the definition of asymptotic stability. The topological reasoning on the "point of infinity" is referred to as the Alexandroff or one-point compactification of $\mathbb{R}^2$.