A2 Introduction to Control Theory: Discrete Time Linear Systems

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Syllabus

Discrete time linear systems. The z-transform and its properties. Conversion between difference equations and z-transform transfer functions. Obtaining the discrete model of a continuous system plus zero order hold from a continuous (Laplace) transfer function. Mapping from s-plane to z-plane. Significance of pole positions and implications to stability. Discrete time system specifications. Discrete time state space systems. Discrete time solutions. Euler discretization.

Lecture notes

These lecture notes are provided as handouts. These notes as well as the lecture slides that follow the same structure are also available on the web (Canvas).

The first four chapters of the notes follow in part lecture material that was produced and was previously taught by Prof. Mark Cannon. His input and his permission to use material from his lecture notes is gratefully acknowledged.

Any comments or corrections shall be sent to `kostas.margellos@eng.ox.ac.uk`

Recommended text

- G F Franklin, J D Powell & M Workman *Digital Control of Dynamic Systems* 3rd edition, Addison Wesley, 1998.

Other reading

- R C Dorf & R H Bishop *Modern Control Systems* Pearson Prentice Hall, 2008
- K J Astrom & R M Murray *Feedback Systems: An Introduction for Scientists and Engineers* Princeton University Press, 2008

The course follows the first book above which will be referred to as [Franklin et al, 1998]; reference to particular book chapters and sections is provided throughout the notes. Consulting the book for detailed elaboration on several concepts as well as for additional examples and exercises is highly recommended.

Contents

1 Introduction

1.1 Digitization*

Discrete time systems are dynamic systems whose inputs and outputs are defined at discrete time instants. Within a control context, digitization is the process of converting a continuous controller (Figure 1) into a set of difference equations that can be implemented by a computer within a digital control system (Figure 2). The digital controller operates on *samples* of the sensed output, resulting upon analogue to digital (ADC) conversion, rather than the continuous signal $y(t)$. The generated discrete time command is then applied to the plant upon digital to analogue (DAC) conversion. This provides practical advantages in terms of accuracy and noise rejection, and simplifies monitoring and simultaneous control of large numbers of feedback loops. However the ability of a digital control system to achieve specified design criteria depends on the choice of digitization method and sample rate.

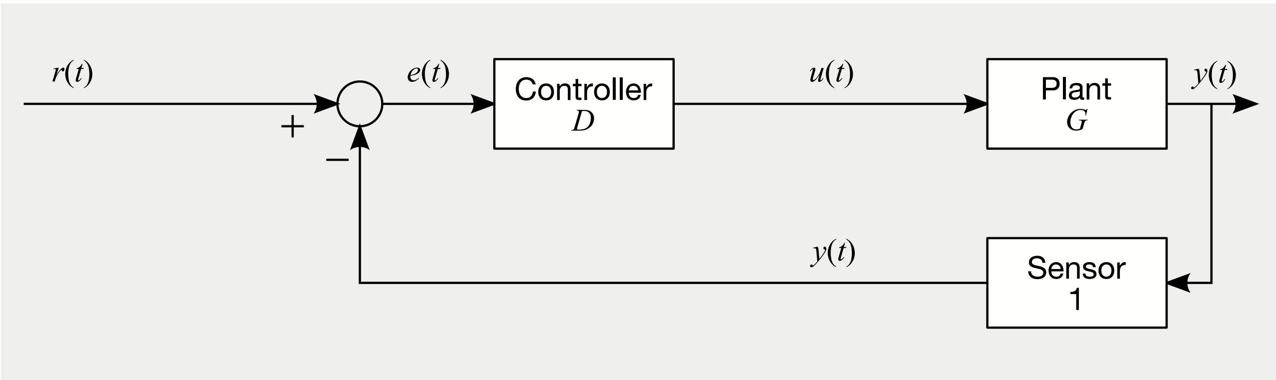


Figure 1: Continuous controller.

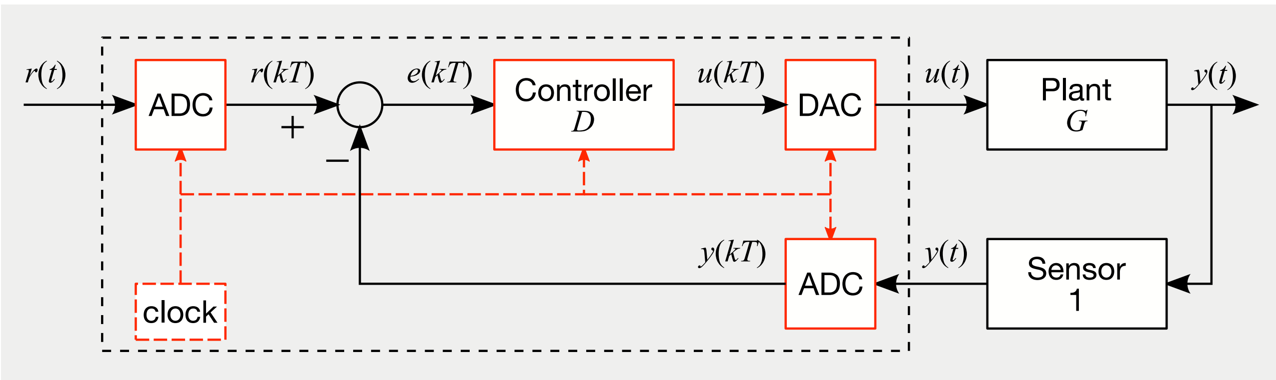


Figure 2: Digital Controller. The dashed box contains sampled signals.

*[Franklin et al, 1998] §3.1

1.2 Motivating examples

One motivating example involves an autonomous racing platform for miniaturized cars. For more details regarding this testbed the reader is referred to the set-up developed at ETH Zurich (see upper panel of Figure 3), that motivated the recent racing car platform that is now built at University of Oxford (see lower panel of Figure 3). The platform consists of a race track, an infrared camera based tracking system and miniature dnano RC cars. It allows for high-speed, real-time control algorithm testing.

Figure 3: Upper panel: RC racing car platform at ETH Zurich (figure taken from `http://control.ee.ethz.ch/∼ racing///`); Lower panel: RC racing car platform at University of Oxford.

The general architecture underpinning the operation of each car is shown in Figure 4. It is based in the following sequence of steps:

1. The camera based vision system captures the cars on the track, each of them characterised by a unique marker pattern.

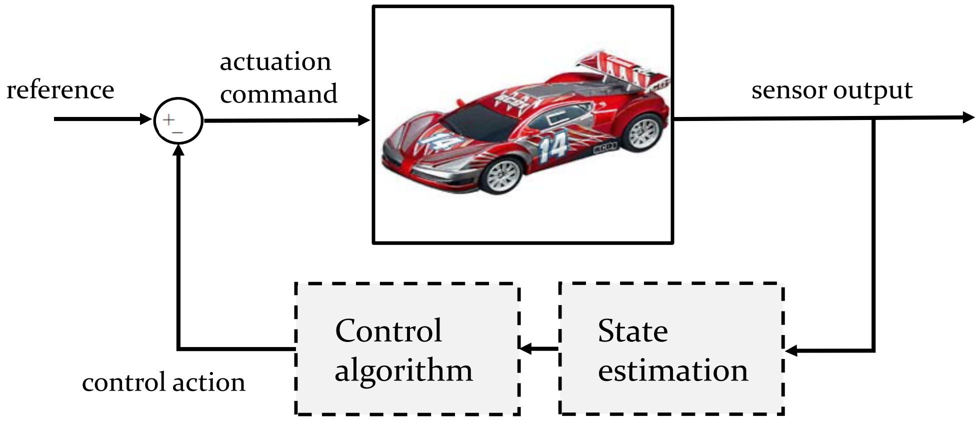


Figure 4: Block diagram of control architecture for each racing car.

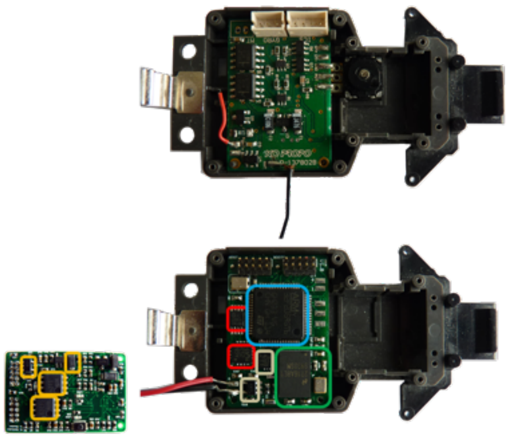


Figure 5: Embedded board of each car (figure taken from `http://control.ee.ethz.ch/~ racing///`).

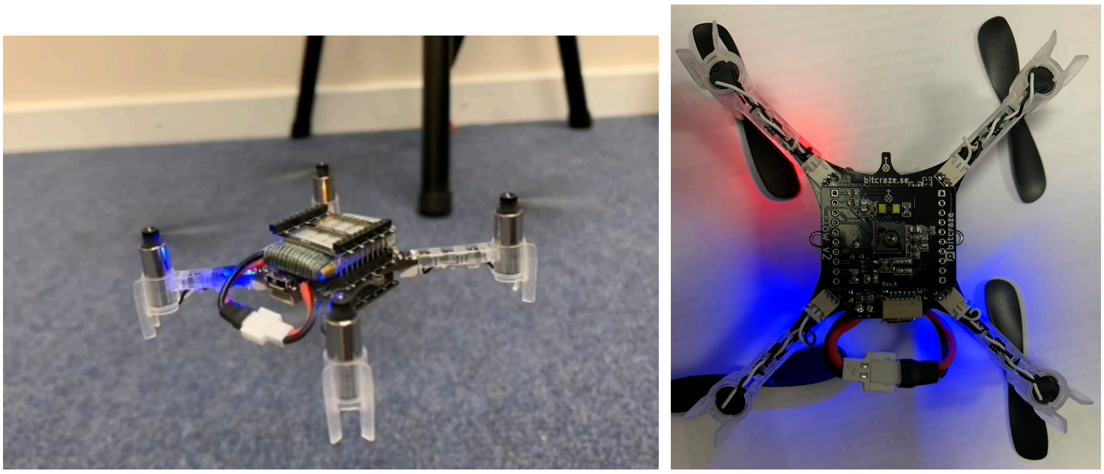


Figure 6: Left panel: Crazyflie arena; Right panel: On board controller.

2. The position and velocity of each car is estimated by means of some state estimation algorithm, and is broadcasted to the computer used for control calculation.

3. The control inputs (e.g., speed commands) are sent via Bluetooth to the

embedded board microcontroller of each car (see Figure 5), which then drives around the track.

An additional example involves the CrazyFlies arena, recently developed at the University of Oxford (see Figure 6). The underlying structure and interplay between the continuous dynamics and discrete logic of the on board controller follows the same rationale with the racing car platform.

The overall configuration in both cases is naturally a discrete system, e.g., see on board microcontroller. The controller could either be designed in continuous time (using e.g., methods from the first 8 lectures of A2 Introduction to Control Theory course) using a continuous time model of car, sampled and then implemented approximately in discrete time, or it could be designed directly as a discrete system using a discrete time model of the car. In the sequel we will derive methods that allow us to analyse both alternatives.

1.3 Organization of the notes

In these notes we will study the entire feedback interconnection of Figure 2: *Chapter 2* will introduce discrete time linear systems and the notion of sampling continuous-time signals. It will focus on the controller block of Figure 2, and provide the machinery to represent it in discrete-time, introducing the z-transform and the discrete-time transfer function (similarly to the continuous-time one). *Chapter 3* will concentrate on obtaining a discrete-time model (transfer function) of the DAC-Plant-Sensor-ADC interconnection, while *Chapter 4* will focus on the closed-loop system, and in particular on how we can investigate its stability properties, analyse its response, and verify whether certain performance criteria are met by the designed controller. *Chapter 5* will introduce the the state space formalism for discrete time systems, and analyze the structure of their solution in the time-domain, thus complementing the transfer function developments. It will also discuss the effects of Euler discretization on approximating differential with difference equations. Finally, the *Appendix* provides a more rigorous treatment and proof of the inverse z-transform.

2 Discrete time linear systems and transfer functions

2.1 Sampling and discrete time systems

Discrete time signals typically emanate from continuous time ones by a procedure called sampling. If a digital computer is used for this purpose, then we typically sample the continuous time signal at a constant rate:

T = sample period (or sampling interval);

$1/T$ = sample rate (or sampling frequency) in Hz;

[$2\pi/T$ = sample rate in rad s^{-1}].

By $t_k = kT$ for $k = \{\ldots -2, -1, 0, 1, 2, 3, \ldots\}$ we denote the *sampling instants* or *sample times*, at which we obtain a "snapshot" of the continuous time signal. Sampling a continuous signal produces a *sampled signal*:

$$y(t) \xrightarrow{\text{sample}} y(kT),$$

where we refer to $y(kT)$ as a *discrete signal* and we will be interchangeably writing it in one of the following ways:

$$y(kT) = y(k) = y_k.$$

Such a signal is the outcome of the analogue to digital conversion (ADC) taking place at the sensor of our system (see Figure 2). With reference to the same figure, at the other end of the process the output of the digital controller $u(kT)$ must be converted back to a continuous time signal. This is usually done by means of a digital to analogue converter (DAC) and a hold circuit which holds the output signal constant during the sampling interval. This is known as a zero-order hold (ZOH); see also Figure 7.

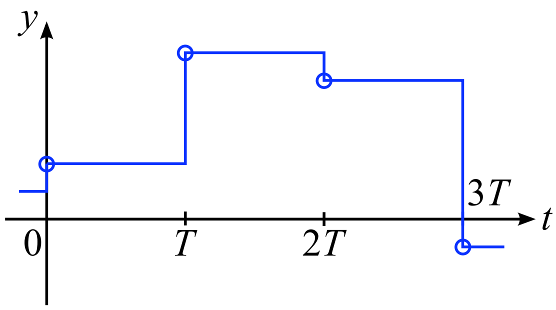


Figure 7: The piecewise constant signal produced by the ZOH.

Discrete time systems are systems whose inputs and outputs are discrete time signals. Due to this interplay of continuous and discrete components, we can observe two discrete time systems in Figure 2, i.e., systems whose input and output are both discrete time signals. The obvious one refers to the controller block, whose input is the error signal e_k and its output is the actuation command u_k. A second less obvious discrete time system is the one that admits u_k as the input and y_k as its output. It has the DAC and ADC converters as well as the plant dynamics that are in continuous time intervened, but still it is a discrete time system. We refer to the latter as a *sampled data system*. Taking the controller block as an example, to derive the corresponding discrete time system suppose we have a access to the transfer function of the controller, namely,

$$D(s) = \frac{U(s)}{E(s)} = \frac{K(s+a)}{(s+b)}.$$

We first unravel this as

$$(s+b)U(s) = K(s+a)E(s),$$

and determine the corresponding differential equation with s replaced by d/dt, i.e.,

$$\frac{du}{dt} + b\,u = K\Big(\frac{de}{dt} + a\,e\Big).$$

Definition 1 (Euler's forward and backward approximation)**.** *We define Euler's forward approximation to the first order derivative by*

$$\frac{du}{dt} \approx \frac{u_{k+1} - u_k}{T} \quad \text{and} \quad \frac{de}{dt} \approx \frac{e_{k+1} - e_k}{T}.$$

Similarly, we define Euler's backward approximation to the first order derivative by

$$\frac{du}{dt} \approx \frac{u_k - u_{k-1}}{T} \quad \text{and} \quad \frac{de}{dt} \approx \frac{e_k - e_{k-1}}{T}.$$

Here we use Euler's forward approximation in out differential equation to obtain the difference equation

$$\frac{u_{k+1} - u_k}{T} + b\,u_k = K\Big(\frac{e_{k+1} - e_k}{T} + a\,e_k\Big).$$

Finally, to implement the controller we need the new control value, u_{k+1},

$$u_{k+1} = (1 - bT)u_k + Ke_{k+1} + K(aT - 1)e_k.$$

Normally T, K, a and b are fixed, so what the computer has to calculate every cycle can be encoded by a standard form recursion

$$u_{k+1} = -a_1 u_k + b_0 e_{k+1} + b_1 e_k.$$

It should be noted that the coefficients of the difference equation change with T, hence the sample rate is usually kept fixed; the smaller T the more the discrete system approximates the continuous one (depending on the Euler approximation used we could formally quantify conditions so that the discrete approximation is at least not divergent). The obtained recursion provides an input-output representation of the discrete time system. In particular, the output of the system u_{k+1} depends on the past output u_k, as well as on the current and past inputs e_{k+1} and e_k, respectively. Since this dependency is linear, we refer to such systems as discrete linear systems.

2.2 The z-Transform

In continuous (linear) systems the Laplace transform was used to transform differential equations to algebraic ones, since the latter were easier to use in view of obtaining input-output relationships, and eventually transfer functions. Similarly, for discrete systems, we will define a new transform, the so called *z-transform*, that will allow us to transform difference or recurrence equations to algebraic ones and construct discrete transfer functions that will allow us to analyze discrete time systems from an input-output point of view.

Definition 2 (z-Transform)**.** *The z-transform $E(z)$ of a discrete signal $e(kT)$ (i.e., $\{e_0, e_1, \dots\}$) is defined by*

$$\begin{aligned} E(z) &= \mathcal{Z}\{e(kT)\} = \mathcal{Z}\{e_k\} \\ &= \sum_{k=0}^{\infty} e(kT)\, z^{-k} = \sum_{k=0}^{\infty} e_k\, z^{-k}. \end{aligned}$$

Note: There are two differences with the z-transform definition of [Franklin et al, 1998], that we will be, however, adopting for this course:

1. In [Franklin et al, 1998] the "two-sided" z-transform is used, with the index k

starting at $-\infty$ whereas these notes (along with HLT) use the "single-sided" z-transform to correspond to the single sided Laplace transform that you are already familiar with. *In this course, all signal values are defined to be zero for $k < 0$ so the two definitions yield the same results*.

2. In [Franklin et al, 1998] the z-transform definition was based on the assumption that upper and lower bounds, respectively, on the magnitude $|z|$ exist, thus ensuring convergence of the series in the z-transform definition. Here, we assume throughout that this is the case, and we will impose such bounds on a case by case basis by exploiting convergence conditions for geometric series as these are involved in the z-transform definition (see example below).

Example 1. *Sample a decaying exponential signal: $x(t) = Ce^{-at}\mathcal{U}(t)$, where $\mathcal{U}(t)$ is the unit step function, to give $x_k = Ce^{-akT}$, for $k \geq 0$.*
Then

$$\begin{aligned} X(z) &= \sum_{k=0}^{\infty} x_k \, z^{-k} \\ &= C \sum_{k=0}^{\infty} e^{-akT} z^{-k} = C \sum_{k=0}^{\infty} (e^{-aT} z^{-1})^k. \end{aligned}$$

This is a geometric series and has a closed form solution if $|e^{-aT} z^{-1}| < 1$, or equivalently if $|z| > e^{-aT}$:

$$X(z) = \frac{C}{1 - e^{-aT} z^{-1}} = \frac{Cz}{z - e^{-aT}} \ .$$

Example 2. *Suppose we have a sequence e_k such that:*

$$e_0, e_1, e_2, e_3, e_4, \ldots = 1.5, 1.6, 1.7, 0, 0, \ldots$$

then delaying e by a period T will create a new sequence (let's call it f_k) with

$$f_0, f_1, f_2, f_3, f_4, \ldots = 0, 1.5, 1.6, 1.7, 0, \ldots$$

Using the above definition of the z-transform gives:

$$\begin{aligned} E(z) &= \sum_{k=0}^{\infty} e_k\, z^{-k} \\ &= 1.5 + 1.6z^{-1} + 1.7z^{-2}, \\ \textit{and} \quad F(z) &= \sum_{k=0}^{\infty} f_k\, z^{-k} \\ &= 1.5z^{-1} + 1.6z^{-2} + 1.7z^{-3} \\ &= z^{-1}(1.5 + 1.6z^{-1} + 1.7z^{-2}) = z^{-1}E(z) \end{aligned}$$

This is an example of a z-transform property that holds true more generally, i.e., if a signal is delayed by a period T, its z-transform gets multiplied by z^{-1}.

Example 3. *Consider the exponential signal*

$$x(t) = Ce^{-at}\mathcal{U}(t),$$

where $\mathcal{U}(t)$ *is the unit step function, delayed by* T *giving rise to*

$$y(t) = Ce^{-a(t-T)}\mathcal{U}(t-T) \ ,$$

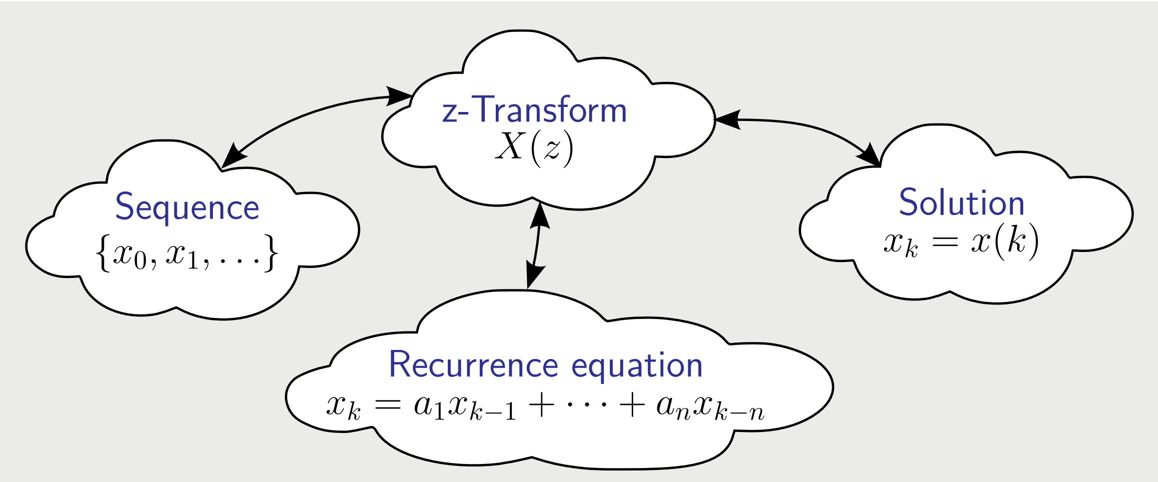
then $y_k = Ce^{-a(k-1)T}$ *for* $k \geq 1$. *Therefore, the z-transform of* y_k *is*

$$\begin{aligned} Y(z) &= \sum_{k=0}^{\infty} y_k\, z^{-k} = \sum_{k=1}^{\infty} Ce^{-a(k-1)T} z^{-k} \\ &= Cz^{-1} \sum_{k=1}^{\infty} (e^{-aT} z^{-1})^{k-1} \\ &= Cz^{-1} \sum_{j=0}^{\infty} (e^{-aT} z^{-1})^{j} = z^{-1}X(z) \end{aligned}$$

which gives $Y(z) = \dfrac{C}{z - e^{-aT}}$.

Z-transform overview

To summarise: $X(z)$ provides an easy way to convert between sequences, recurrence equations and their closed-form solutions.



2.3 Properties of the z-transform*

Let $F(z) = \mathcal{Z}\{f(kT)\} = \sum_{k=0}^{\infty} f_k\, z^{-k}$ and $G(z) = \mathcal{Z}\{g(kT)\} = \sum_{k=0}^{\infty} g_k\, z^{-k}$ be the z-transform of f and g, respectively. We then have the following properties:

1. Time delay: $\mathcal{Z}\{f(kT - nT)\} = z^{-n}F(z)$, for $n > 0$.

Proof: We have that

$$\begin{aligned}\mathcal{Z}\{f(kT - nT)\} &= \sum_{k=0}^{\infty} f_{k-n} z^{-k} = \sum_{k=n}^{\infty} f_{k-n} z^{-k} \\ &= \sum_{k=0}^{\infty} f_k z^{-(k+n)} = z^{-n} \sum_{k=0}^{\infty} f_k z^{-k} = z^{-n} F(z),\end{aligned}$$

where the first equality follows from the fact that for all $k < n$, f_{k-n} is assumed to be zero, and the second inequality follows by a change of the summation index. Note that in [Franklin et al, 1998] no constraint on n is imposed, since the two-sided z-transform is employed.

2. Linearity: $\mathcal{Z}\{\alpha f(kT) + \beta g(kT)\} = \alpha F(z) + \beta G(z)$.

Proof: We have that

$$\begin{aligned}\mathcal{Z}\{\alpha\, f(kT) + \beta g(kT)\} &= \sum_{k=0}^{\infty} (\alpha f_k + \beta g_k) z^{-k} \\ &= \alpha \sum_{k=0}^{\infty} f_k z^{-k} + \beta \sum_{k=0}^{\infty} g_k z^{-k} = \alpha F(z) + \beta G(z).\end{aligned}$$

*[Franklin et al, 1998] §4.6

3. Differentiation: $\mathcal{Z}\big\{k\, f(kT)\big\} = -z\dfrac{d}{dz}\mathcal{Z}\big\{f(kT)\big\}.$

Proof: We have that

$$-z\frac{d}{dz}\mathcal{Z}\big\{f(kT)\big\} = -z\frac{d}{dz}\sum_{k=0}^{\infty} f_k z^{-k}$$
$$= -z\sum_{k=0}^{\infty}(-k) f_k z^{-k-1} = \sum_{k=0}^{\infty} k f_k z^{-k} = \mathcal{Z}\big\{k\, f(kT)\big\}.$$

4. Convolution: $\mathcal{Z}\left\{\sum_{i=0}^{\infty} f(iT)\, g(kT - iT)\right\} = F(z)\, G(z).$

Proof: We have that

$$\mathcal{Z}\left\{\sum_{i=0}^{\infty} f(iT)\, g(kT - iT)\right\} = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty} f_i\ g_{k-i}\ z^{-k} = \sum_{p=0}^{\infty}\sum_{i=0}^{\infty} f_i\ g_p\ z^{-(p+i)}$$
$$= \sum_{i=0}^{\infty} f_i z^{-i}\ \sum_{p=0}^{\infty} g_p z^{-p} = F(z)G(z),$$

where the second equality follows from changing $k - i$ to p, and noticing that the summation limits remain unchanged since all terms corresponding to negative indices are assumed to be zero. For the last equality notice that i and p are arbitrary variables so each summation corresponds to the z-transform of f and g, respectively.

5. Final value theorem: $\lim_{k\to\infty} f(kT) = \lim_{z\to 1}\big\{(z-1)F(z)\big\}$, if the poles of $(z-1)F(z)$ are inside the unit circle and $F(z)$ converges for all $|z| > 1$.

Proof: We have that

$$\begin{aligned}
(z-1)F(z) &= (z-1)\sum_{k=0}^{\infty} f_k z^{-k} \\
&= \sum_{k=0}^{\infty} f_k z^{-(k-1)} - \sum_{k=0}^{\infty} f_k z^{-k} \\
&= f_0 z + \sum_{k=1}^{\infty} f_k z^{-(k-1)} - \sum_{k=0}^{\infty} f_k z^{-k} \\
&= f_0 z + \sum_{k=0}^{\infty} f_{k+1} z^{-k} - \sum_{k=0}^{\infty} f_k z^{-k} \\
&= f_0 z + \lim_{K\to\infty}\sum_{k=0}^{K}(f_{k+1} - f_k) z^{-k},
\end{aligned}$$

where the fourth equality follows from the third one by changing the summation index of the first summation, and the last one constitutes an equivalent way of representing a summation limit that tends to infinity.

Take now in both sides the limit as z tends to one. We then have that

$$\begin{aligned}\lim_{z\to 1}(z-1)F(z) &= \lim_{z\to 1}\Big(f_0 z + \lim_{K\to\infty}\sum_{k=0}^{K}(f_{k+1}-f_k)z^{-k}\Big),\\ &= f_0 + \lim_{K\to\infty}\sum_{k=0}^{K}(f_{k+1}-f_k)\\ &= f_0 + \lim_{K\to\infty} f_{K+1} - f_0 = \lim_{k\to\infty} f(kT),\end{aligned}$$

where the second equality follows from the first one by exchanging the order of limits and taking the limit as $z \to 1$. The third equality is due to the fact that $\sum_{k=0}^{K}(f_{k+1}-f_k)$ is a so called telescopic series, with all intermediate terms cancelling each other.

It should be noted that the fact that $F(z)$ converges for all $|z| > 1$ justifies our assumption that the summation of terms corresponding to negative indices is zero and ensures that $\lim_{k\to\infty} f_k$ exists, while the fact the poles of the poles of $(z-1)F(z)$ are inside the unit circle ensures that taking the limit with respect to z is well defined, and the exchange of limits in the previous derivation is justified. Such a condition will be discussed in more detail in Chapter 5. Note that in [Franklin et al, 1998], §4.6.1, an alternative proof is provided.

2.4 The discrete transfer function*

We have already seen that in a typical controller the new control output is calculated as a function of the new error, plus previous values of error and control output. In general we have a linear recurrence equation (also often called a linear difference equation),

$$\begin{aligned}u_k &= -a_1 u_{k-1} - a_2 u_{k-2} - \cdots - a_{k-n}u_{k-n} + b_0 e_k + b_1 e_{k-1} + \cdots + b_m e_{k-m}\\ &= -\sum_{i=1}^{n} a_i u_{k-i} + \sum_{j=0}^{m} b_j e_{k-j}.\end{aligned}$$

*[[Franklin et al, 1998] et al, 1998] §4.2

Let $U(z)$ and $E(z)$ denote the z-transforms of u_k and e_k, respectively. Using the linearity and time delay properties of the z-transform we have that

$$U(z) = -a_1 z^{-1} U(z) - a_2 z^{-2} U(z) - \cdots + b_0 E(z) + b_1 z^{-1} E(z) + \cdots$$

Rearranging terms, we obtain the *z-transform transfer function*:

$$D(z) = \frac{U(z)}{E(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots}\ .$$

We can turn the transfer function into a rational function of z by multiplying numerator and denominator by z^n (so that the last coefficient in the denominator is a_n):

$$D(z) = \frac{b_0 z^n + b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n}.$$

Definition 3 (Discrete transfer function)**.** *The z-transform transfer function is given in a factorized form by*

$$D(z) = b_0 \frac{\Pi_{j=1}^{m}(z - z_j)}{\Pi_{i=1}^{n}(z - p_i)} z^{n-m},$$

assuming $b_0 \neq 0$*. Similarly to the continuous case,* z_j*,* $j = 1, \ldots, m$*, are said to be the zeros and the* p_i*,* $i = 1, \ldots, n$*, are the poles of* $D(z)$*, and occur either as real numbers or in complex conjugate pairs.*

Clearly when $n > m$, there are $(n - m)$ zeros at the origin and when $n < m$, there are $(m - n)$ poles at the origin. Like continuous systems we must have at least as many poles as zeros*. If $b_0 \neq 0$, we have an equal number of each and if $b_0 = 0$, we have one fewer zero. Likewise if $b_0 = b_1 = 0$ we have two fewer zeros, etc.

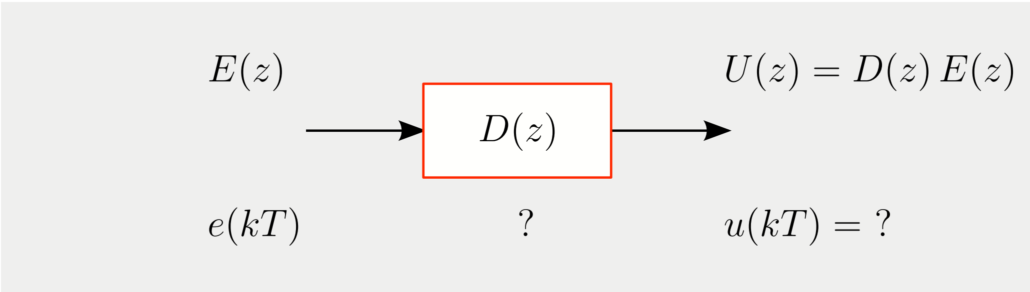
By defining z-domain transfer functions in this way, we can extend to discrete systems all the techniques that were developed for continuous systems in the s-domain. In particular, series or feedback interconnections among transfer functions as often encountered in block-diagrams are addressed identically with the continuous-time case. However, when it comes to stability and performance analysis certain differences are encountered; these are discussed in Chapter 4.

*A greater number of zeros than poles in $D(z)$ would indicate that the system is non-causal since u_k would then depend on e_{k+i}, for $i > 0$.

2.5 Pulse response and convolution

2.5.1 Pulse response

In continuous systems there is a fundamental connection between the transfer function in the frequency domain and the impulse response in the time domain – they constitute a Laplace transform pair. We will investigate the analogous property for discrete systems.



Take the case where $e(kT)$ is the discrete version of the *unit pulse*, i.e.,

$$e_k = \begin{cases} 1 & \text{for } k = 0, \\ 0 & \text{for } k > 0 \end{cases}$$

this can be written as $e_k = \delta_k$, where δ_k is the δ-function.

In this case

$$E(z) = \sum_{k=0}^{\infty} e_k\, z^{-k} = 1 \;\Rightarrow\; U(z) = D(z)\, E(z) = D(z).$$

As we have applied a unit pulse to this system, the output, $u(kT)$ will be its pulse response and $U(z)$ will be the z-transform of the pulse response. This implies that, similarly to the continuous time case,

Fact 1 (Pulse response = transfer function)**.** *The transfer function $D(z)$ is the z-transform of the pulse response.*

Example 4. *Suppose a controller is defined by the recurrence equation:*

$$u_k = u_{k-1} + \frac{T}{2}(e_k + e_{k-1}).$$

After some algebraic manipulation we deduce that the transfer function from

$E(z)$ to $U(z)$ is given by

$$D(z) = \frac{U(z)}{E(z)} = \frac{T}{2}\left(\frac{z+1}{z-1}\right).$$

To verify this, apply a unit pulse, $e_k = \delta_k$ and look at the pulse response (assume as always that $u_k = 0$ for $k < 0$):

k	u_{k-1}	e_k	e_{k-1}	u_k
0	0	1	0	$T/2$
1	$T/2$	0	1	T
2	T	0	0	T
3	T	0	0	T
$\vdots$				$\vdots$

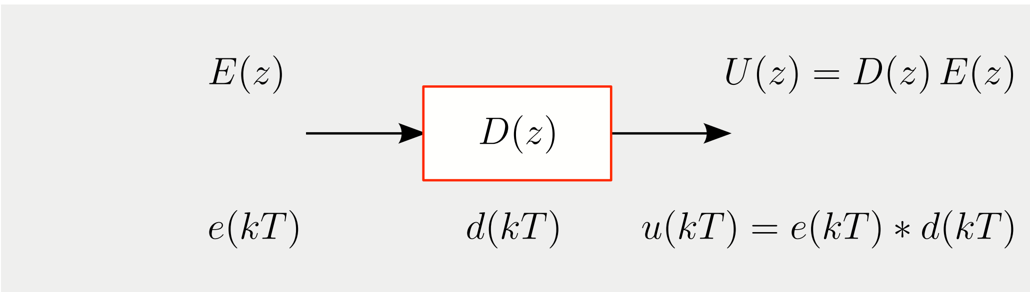
The z-transform of u_k is:

$$\begin{aligned} U(z) &= T/2 + Tz^{-1} + Tz^{-2} + Tz^{-3} + \dots \\ &= \sum_{k=0}^{\infty} T\,z^{-k} - T/2 \\ &= \frac{T}{1-z^{-1}} - \frac{T}{2} = \frac{T}{2}\left(\frac{z+1}{z-1}\right), \qquad \text{[for } |z| > 1\text{]}, \end{aligned}$$

which is equal to $D(z)$ as expected.

2.5.2 Convolution

Consider the following interconnection:



Because the system is linear and because $e(kT)$ is just a sequence of values, for example, $e_0, e_1, e_2, e_3, e_4, \ldots = 1.0, 1.2, 1.3, 0, 0, \ldots$, we can decompose $e(kT)$ into a series of pulses: $e(kT) = 1.0\delta_k + 1.2\delta_{k-1} + 1.3\delta_{k-2}$.

We can now determine the response of $u(kT)$ for each of the individual pulses and simply sum them to get the response when the input is $e(kT)$. Suppose the pulse response of the system is:

$$d_0, d_1, d_2, d_3, d_4, d_5, \ldots = 1, 0.5, 0.3, 0.2, 0.1, 0, \ldots$$

Then we obtain:

k	0	1	2	3	4	5	6	7
$e(kT)$	1	1.2	1.3	0	0	0	0	0
$e(0)\,d(kT)$	1	0.5	0.3	0.2	0.1	0	0	0
$e(1)\,d(kT-T)$	0	1.2	0.6	0.36	0.24	0.12	0	0
$e(2)\,d(kT-2T)$	0	0	1.3	0.65	0.39	0.26	0.13	0
$u(kT)$	1	1.7	2.2	1.21	0.73	0.38	0.13	0

From the table it can be observed that

$$\begin{aligned}
u_0 &= e_0 d_0, \\
u_1 &= e_0 d_1 + e_1 d_0, \\
u_2 &= e_0 d_2 + e_1 d_1 + e_2 d_0, \\
u_3 &= e_0 d_3 + e_1 d_2 + e_2 d_1 + e_3 d_0, \\
&\vdots \\
u_k &= \sum_{i=0}^{k} e_i\, d_{k-i},
\end{aligned}$$

which is the *convolution* of the input and the pulse response (sometimes written as $u_k = e_k * d_k$). A graphical calculation of convolution can be shown in Figure 8. From the last row of the table, it can be shown that the z-transform of $u(kT)$ is given by

$$U(z) = \sum_{k=0}^{\infty} u(kT) z^{-k} = 1 + 1.7z^{-1} + 2.2z^{-2} + \ldots$$

On the other hand, we can easily calculate the z-transforms of $e(kT)$ and $d(kT)$ which are given by

$$\begin{aligned}
E(z) &= \sum_{k=0}^{\infty} e(kT) z^{-k} = 1 + 1.2z^{-1} + 1.3z^{-2} + \ldots \\
D(z) &= \sum_{k=0}^{\infty} d(kT) z^{-k} = 1 + 0.5z^{-1} + 0.3z^{-2} + \ldots
\end{aligned}$$

It can be directly verified, that $U(z) = D(z)E(z)$, as expected due to the convolution property of the z-transform. In general, the following fact holds.

Fact 2 (Prodcust in the z-domain = Convolution in the time domain)**.** *Multiplying the input by the transfer function in the z-domain is equivalent to convolving the input with the pulse response in the time domain.*

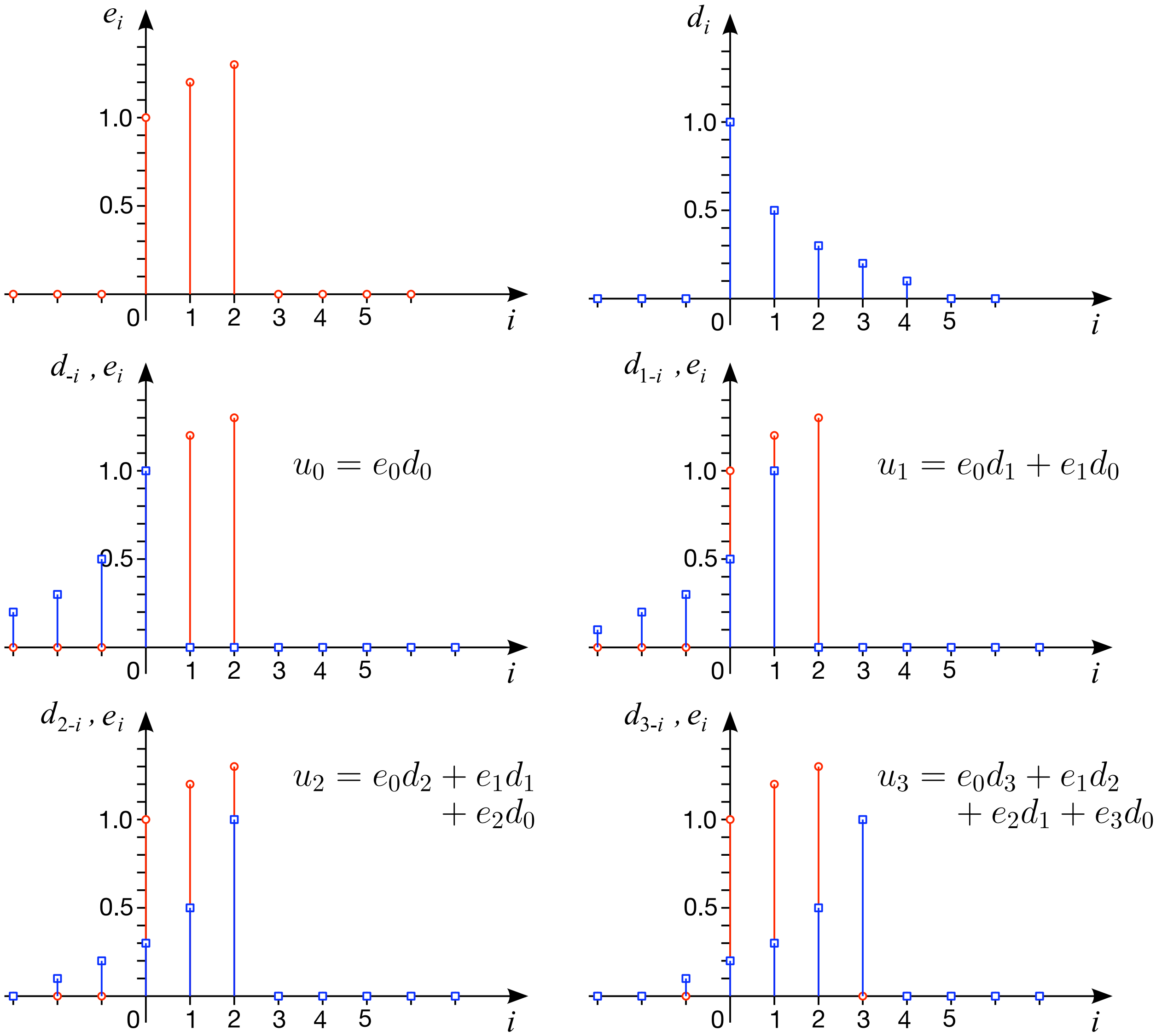


Figure 8: Graphical illustration of the setps required to compute the convolution of two sequences, i.e., e_k and d_k.

It should be noted that for continuous systems, it is usually more convenient to multiply in the frequency domain than to convolve in the time domain (which involves integrating). However for discrete systems the convolution involves only multiplying and adding, so is very easy to perform using a computer.

2.6 Computing the inverse z-transform

We have determined already a few methods to compute the z-transform $Y(z)$ given a sequence y_k:

1. Evaluate the sum $Y(z) = \sum_{k=0}^{\infty} y_k z^{-k}$ directly.

   This method was used earlier to compute the z-transform of a sequence with only a finite number of non-zero terms. It was also the method used to compute the z-transform of exponential signals using the formula for a geometric series.

2. Use standard results in tables (e.g. HLT p. 17).

What has not been discussed yet is how to obtain a discrete time signal if its z-transform is available, or in other words, we have not discussed about the inverse z-transform Some methods of determining the inverse z-transform y_k of a function $Y(z)$ are provided below:

1. Determine the coefficients $y_0, y_1, y_2, \ldots$ of $1, z^{-1}, z^{-2}, \ldots$ by computing the Maclaurin series expansion of $Y(z)$. For example,

   $$\begin{aligned} Y(z) = \frac{1-z^{-1}}{1+z^{-1}} &= 1 - 2\frac{z^{-1}}{1+z^{-1}} = 1 - 2z^{-1}(1 - z^{-1} + z^{-2} - \ldots) \\ &= 1 - 2z^{-1} + 2z^{-2} - \ldots = \mathcal{Z}\{1, -2, 2, -2, \ldots\}, \end{aligned}$$

   so $\{y_0, y_1, y_2, y_3, \ldots\} = \{1, -2, 2, -2, \ldots\}$ or $y_k = 2(-1)^k - \delta_k$.

2. Use standard results in tables (e.g. HLT p. 17). For example,

   $$Y(z) = \frac{z \sin aT}{z^2 - 2z \cos aT + 1} = \mathcal{Z}\{0, \sin(aT), \sin(2aT), \ldots\},$$

   hence $y_k = \sin(akT)$ for $k = 0, 1, \ldots$.

3. Treat $Y(z)$ as a transfer function and determine its pulse response y_k. For example,

   $$Y(z) = \frac{1+2z^{-1}}{1-2z^{-1}} \quad \Longrightarrow \quad y_k - 2y_{k-1} = \delta_k + 2\delta_{k-1} Z \text{ with } \; y_k = 0 \text{ for } k < 0,$$

   hence $\{y_0, y_1, y_2, \ldots\} = \{1, 4, 8, 16, \ldots\}$ or $y_k = 2^{k+1} - \delta_k$, for $k = 0, 1, \ldots$.

4. The inverse z-transform is defined by

$$f(kT) = \mathcal{Z}^{-1}\{F(z)\} = \frac{1}{2\pi j} \oint z^{k-1} F(z)\, dz$$

(where the contour encircles the poles of $F(z)$).

Calculating such an integral can be significantly more difficult compared to the (forward) z-transform (see Appendix for a detailed elaboration). This is also the case with the inverse Laplace transform, where the standard calculation practice is to consider the partial fraction expansion of the transfer function and determine the factors by means of transform tables.

2.7 Summary

This main learning outcomes of the chapter can be summarized as follows:

- Discrete time systems are represented by means of recurrence equations.
- The z-transform of a sequence u_k is defined $U(z) = \sum_{k=0}^{\infty} u_k\, z^{-k}$.
- The z-transform exhibits certain interesting properties: Time delay; Linearity; Differentiation; Convolution; Final value theorem.
- Discrete transfer functions can be determined directly from linear recurrence equations using the z-tranform.
- Discrete transfer functions map the z-transform of the input sequence to the z-transform of the output sequence, e.g., $U(z) = D(z)E(z)$.
- The discrete transfer function of a linear system is the z-transform of the system's pulse response.
- Multiplication in the z-domain is equivalent to convolution in the time-domain, e.g., $u_k = \sum_{i=0}^{\infty} e_i d_{k-i}$.
- Given the z-transform $Y(z)$, different methods were discussed to obtain its inverse y_k.

3 Discrete models of sampled data systems

3.1 Pulse transfer function models*

The z-domain provides a means of handling the discrete part of the system. It remains to deal with the rest of the system, namely the plant, actuators, sensors etc, which operate in continuous time, or in other words computing the transfer function of the so called sampled data system between u_k and y_k with reference to Figure 2. For convenience, this part is shown in Figure 9. To this end, given the continuous part of the system, $G(s)$, driven by a zero-order hold, the problem is to calculate a transfer function† $G(z)$.

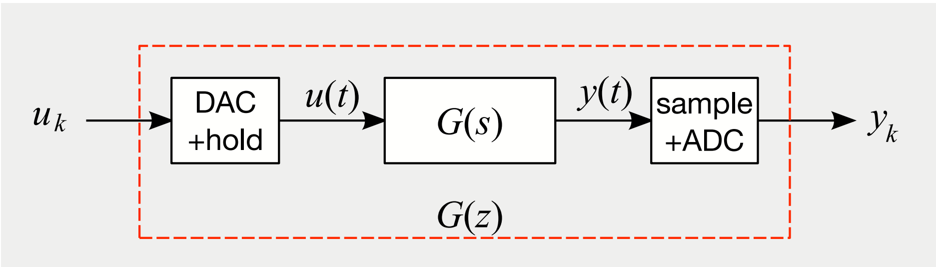


Figure 9: Sampled data system.

To achieve this, first consider the ZOH. For each sample time k it admits u_k as input and has the continuous time signal $u(t)$ (rectangular pulse of height u_k and width T) as output (see Figure 10).

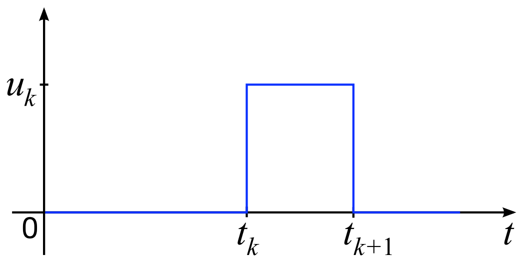


Figure 10: Zero-order hold impulse response

For each sample u_k, the output of the ZOH is therefore a step of height u_k at time $t_k = kT$ plus a step of height $-u_k$ at time $t_{k+1} = (k+1)T$, so the continuous

*Franklin §4.3

†We denote the continuous and discrete transfer functions of a sampled data system G as $G(s)$ and $G(z)$ respectively. These functions of s and of z will not generally be the same – whether we're talking about the continuous or discrete model is indicated by whether the argument is s or z.

signal at the output of the ZOH has the Laplace transform:

$$\mathcal{L}\Big\{u_k\,.\,\mathcal{U}(t-kT)\Big\} - \mathcal{L}\Big\{u_k\,.\,\mathcal{U}\big(t-(k+1)T\big)\Big\} = e^{-kTs}\,\frac{u_k}{s} - e^{-(k+1)Ts}\,\frac{u_k}{s}$$
$$= u_k\,\frac{(1-e^{-Ts})}{s}\,e^{-kTs},$$

where $\mathcal{U}(t)$ is the unit step function at $t=0$. By means of superposition, considering samples for $k=0,1,\ldots$, the Laplace transform of the output of ZOH is given by

$$U(s) = \sum_{k=0}^{\infty} u_k\,\frac{(1-e^{-Ts})}{s}\,e^{-kTs}.$$

However, in Section 2.5, it was shown that the pulse response of a system coincides with its discrete time transfer function, i.e., $Y(z)=G(z)$ when $u_k=\delta_k$. Therefore, to calculate $G(z)$ it suffices to set $u_k=\delta_k$ and compute $Y(z)$.

To this end, we can follow a five-step process to compute $G(z)$:

1. **Compute $Y(s)$ for $u_k=\delta_k$.** Consider the Laplace transfer function of the output, which is given by $G(s)U(s)=\sum_{k=0}^{\infty} u_k\,\frac{(1-e^{-Ts})G(s)}{s}\,e^{-kTs}$. Setting $u_k=\delta_k$, all terms in the summation become zero except from the term corresponding to $k=0$. Hence,
$$Y(s) = (1-e^{-Ts})\frac{G(s)}{s}.$$
2. **Compute the continuous time output $y(t)$.** Taking the inverse Laplace transform of $Y(s)$,
$$y(t) = \mathcal{L}^{-1}\Big\{\frac{(1-e^{-Ts})G(s)}{s}\Big\}.$$
3. **Compute the discrete time output y_k by sampling $y(t)$.**
$$y_k = \bigg\{\mathcal{L}^{-1}\Big\{\frac{(1-e^{-Ts})G(s)}{s}\Big\}_{t=kT}\bigg\}.$$
4. **Compute the z-Transform $G(z)$.** Compute the z-Transform $Y(z)$ of y_k, and recall that for the pulse response we have that $G(z)=Y(z)$. Hence,
$$G(z) = \mathcal{Z}\bigg\{\mathcal{L}^{-1}\Big\{\frac{(1-e^{-Ts})G(s)}{s}\Big\}_{t=kT}\bigg\}.$$
5. **Simplify $G(z)$.** Recall that e^{-Ts} represents a delay of T, which is equivalent to z^{-1} in the z-transform domain. Therefore, the expression for $G(z)$ can be simplified as summarized below.

The **sampled data system transfer function** is given by

$$G(z) = (1 - z^{-1})\mathcal{Z}\Big\{\mathcal{L}^{-1}\big\{\frac{G(s)}{s}\big\}_{t=kT}\Big\}.$$

For simplicity, this operation can be written as $(1 - z^{-1})\mathcal{Z}\Big\{\frac{G(s)}{s}\Big\}$.

Example 5. *Suppose that* $G(s) = \frac{a}{s+a}$. *Then, using the formula above,* $G(z)$ *is obtained from*

$$\begin{aligned} G(z) &= (1 - z^{-1})\mathcal{Z}\Big\{\mathcal{L}^{-1}\big\{\frac{a}{s(s+a)}\big\}\Big\} \\ &= (1 - z^{-1})\mathcal{Z}\Big\{\mathcal{L}^{-1}\big\{\frac{1}{s} - \frac{1}{(s+a)}\big\}\Big\} \\ &= (1 - z^{-1})\mathcal{Z}\Big\{\big\{\mathcal{U}(t) - e^{-at}\mathcal{U}(t)\big\}_{t=kT}\Big\} \\ &= (1 - z^{-1})\mathcal{Z}\big\{\mathcal{U}(kT) - e^{-akT}\mathcal{U}(kT)\big\}. \end{aligned}$$

But the z-transform of the unit step function is

$$\mathcal{Z}\big\{\mathcal{U}(kT)\big\} = \sum_{k=0}^{\infty} z^{-k} = \frac{z}{z-1}, \qquad \text{for } |z| > 1,$$

and similarly the exponentially decaying term has z-transform

$$\mathcal{Z}\big\{e^{-akT}\mathcal{U}(kT)\big\} = \sum_{k=0}^{\infty} e^{-akT} z^{-k} = \frac{z}{z - e^{-aT}}, \qquad \text{for } |z| > e^{-aT}.$$

Hence,

$$G(z) = (1 - z^{-1})\Big(\frac{z}{z-1} - \frac{z}{z - e^{-aT}}\Big) = \frac{1 - e^{-aT}}{z - e^{-aT}} \; .$$

Example 6. *Now suppose that the ADC and DAC in the previous example are not synchronized, and that there is a fixed delay of* τ *seconds between sampling the plant output at the ADC and updating the DAC,* $0 \le \tau \le T$. *This situation arises for example if there is a non-negligible processing delay (known as latency) in the controller. The delay has exactly the same effect as a delay of* τ *in the continuous system, which is represented as a factor*

$e^{-s\tau}$ multiplying $G(s)$. Thus we can determine a new pulse transfer function, $G_\tau(z)$, which represents the same plant as $G(z)$ but now includes the delay of τ:

$$\begin{aligned} G_\tau(z) &= (1 - z^{-1})\mathcal{Z}\Big\{\mathcal{L}^{-1}\Big\{\frac{e^{-s\tau}a}{s(s+a)}\Big\}\Big\} \\ &= (1 - z^{-1})\mathcal{Z}\big\{\mathcal{U}(kT-\tau) - e^{-a(kT-\tau)}\mathcal{U}(kT-\tau)\big\}. \end{aligned}$$

Substituting for

$$\begin{aligned} \mathcal{Z}\big\{\mathcal{U}(kT-\tau)\big\} &= \sum_{k=1}^{\infty} z^{-k} = \frac{1}{z-1}, \\ \mathcal{Z}\big\{e^{-a(kT-\tau)}\mathcal{U}(kT-\tau)\big\} &= \sum_{k=1}^{\infty} e^{-a(kT-\tau)} z^{-k} = \frac{e^{-a(T-\tau)}}{z - e^{-aT}}, \end{aligned}$$

(recall that $\tau < T$ so the impulse response is delayed by at most one sample), and rearranging gives

$$G_\tau(z) = z^{-1}\,\frac{(1 - e^{-a(T-\tau)})z + (e^{-a(T-\tau)} - e^{-aT})}{z - e^{-aT}}.$$

Notice that if $\tau = T$, then $G_\tau(z) = z^{-1}G(z)$, as expected.

Pulse transfer functions of systems with several outputs

A discrete transfer function can only describe the map from one discrete signal to another. This is important to remember when calculating the pulse transfer functions of systems with more than one output variable.

Consider for example the system in Figure 11, which has two output variables: y and z. The pulse transfer function between u_k and y_k is

$$\frac{Y(z)}{U(z)} = (1 - z^{-1})\mathcal{Z}\Big\{\frac{G_1(s)}{s}\Big\},$$

whereas that from u_k to z_k is clearly

$$\frac{Z(z)}{U(z)} = (1 - z^{-1})\mathcal{Z}\Big\{\frac{G_1(s)G_2(s)}{s}\Big\}.$$

Even though z is related to y via $Z(s) = G_2(s)Y(s)$, this cannot be used directly to simplify computation of the the pulse transfer function from u_k to z_k because z

and y are related by a purely continuous system, and in particular

$$(1-z^{-1})\mathcal{Z}\Big\{\frac{G_1(s)G_2(s)}{s}\Big\} \neq (1-z^{-1})\mathcal{Z}\Big\{\frac{G_1(s)}{s}\Big\}\,\mathcal{Z}\Big\{G_2(s)\Big\}\,.$$

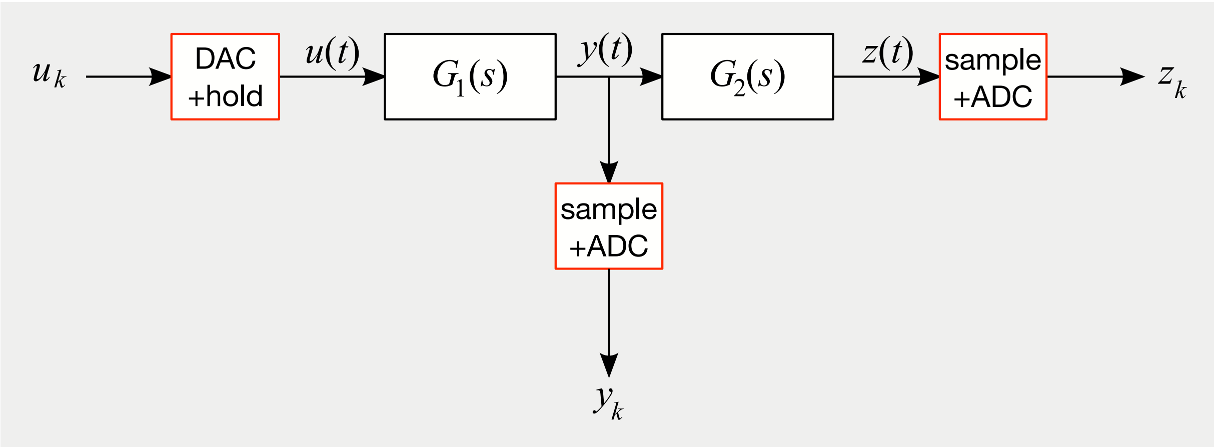


Figure 11: Sampled data system with two outputs.

3.2 The ZOH transfer function

In the previous section DAC and ZOH were considered together. We now focus on their individual effects on the signals involved, as shown in Figure 12. To this end, recall that the Laplace transform of the signal at the output of the ZOH is given by $U(s) = \sum_{k=0}^{\infty} u_k e^{-kTs}(1-e^{-Ts})/s$. However, $u^*(t)$ is a train of Dirac functions, hence $U^*(s) = \sum_{k=0}^{\infty} u_k e^{-kTs}$.

As $U(s) = ZOH(s)U^*(s)$, by a direct comparison between the expressions of $U(s)$ and $U^*(s)$, we obtain

$$ZOH(s) = \frac{1-e^{-Ts}}{s}.$$

This suggests that the DAC conversion process can be represented mathematically by two operations: construction of U^* from the discrete input signal u_k; and, then filtering by $ZOH(s) = (1-e^{-sT})/s$.

Using the inverse Laplace transform we have that

$$u^*(t) = \mathcal{L}^{-1}\big\{U^*(s)\big\} = \mathcal{L}^{-1}\Big\{\sum_{k=0}^{\infty} u_k e^{-kTs}\Big\} = \sum_{k=0}^{\infty} u_k\,\delta(t-kT)\,,$$

i.e., a train of δ-functions. The time-domain signal $u(t)$ after the ZOH is the filtered version of u^* shown in Figure 12. The signals in the output are derived based on a similar reasoning.

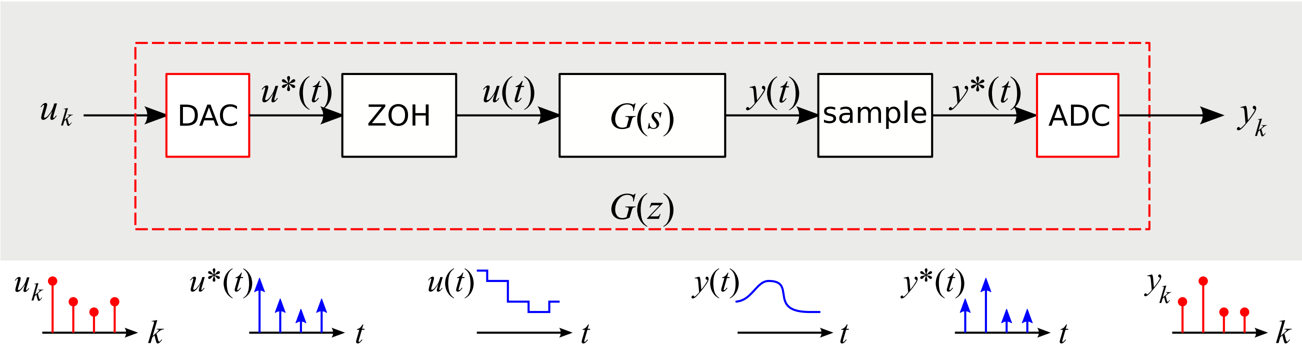


Figure 12: Impulse modulation model of sampled signals in a sampled data system.

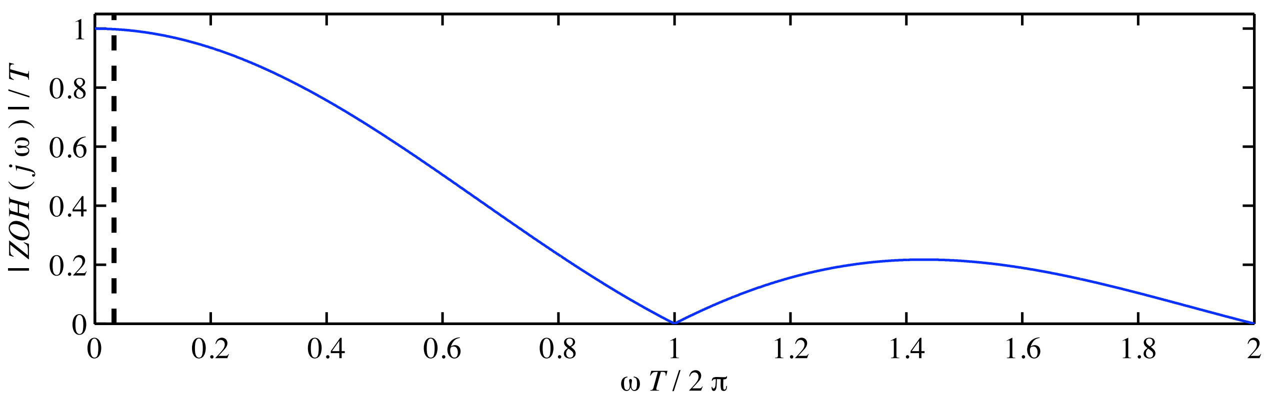


Figure 13: The gain of the ZOH frequency response, $|ZOH(j\omega)|$.

We can compute the ZOH frequency response, which leads to

$$\begin{aligned} ZOH(j\omega) = \frac{(1-e^{-j\omega T})}{j\omega} &= Te^{-j\omega T/2}\frac{(e^{j\omega T/2}-e^{-j\omega T/2})}{j\omega T} \\ &= Te^{-j\omega T/2}\frac{\sin(\omega T/2)}{\omega T/2}, \end{aligned}$$

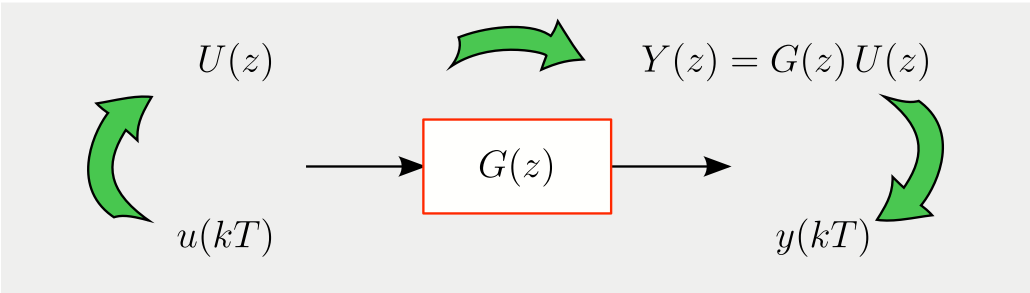
which has phase $\arg\{ZOH(j\omega)\} = -\omega T/2$, and gain

$$|ZOH(j\omega)| = T\left|\frac{\sin(\omega T/2)}{\omega T/2}\right| = T|\mathrm{sinc}(\omega T/2)|.$$

Therefore, the ZOH introduces a phase lag of $-\omega T/2$, and its magnitude follows a sinc function (Figure 13). It can be observed that the magnitude remains approximately constant for $\omega \leq \frac{\pi}{30T}$ (dashed line in Figure 13), while for that value the phase lag becomes negligible, i.e., $\frac{\pi}{30} \approx 6°$. This observation suggests that for the effect of the ZOH to be negligible as far as reproducing a signal close to the original continuous one is concerned, the sampling rate $1/T$ should be quite high (at least 30 times the bandwith for bandlimited signals). Note that this should not be confused with the Nyquist's sampling theorem, which requires sampling with frequency at least two times the bandwidth (for bandlimited signals) to avoid aliasing.

3.3 Signal analysis and dynamic response*

The z-transform can be used to compute the response of a discrete system analogously to the way that Laplace transforms are used for continuous systems:



1. Compute the transfer function $G(z)$ of the system.
2. Compute the transform $U(z)$ of the input signal $u(kT)$ (or look it up in tables).
3. Multiply the transform of the input by the transfer function to find the transform of the output $Y(z)$.
4. Invert this transform to obtain the output $y(kT)$ (or look it up in tables).

3.4 Laplace and z-transform of commonly encountered signals

The following two tables provide the z-transforms for several useful signals. It should be noted that:

1. All of these z-transforms (and Laplace transforms) apply to *signals* f where $F(z) = \mathcal{Z}\{f(kT)\}$ (and $\mathcal{F}(s) = \mathcal{L}\{f(t)\}$).
2. In all cases, the functions of t are zero for $t < 0$.
3. The closed form solutions for $F(z)$ are valid for $|z| > |p_i|$, where p_i are their pole positions.

*[Franklin et al, 1998] §4.4

4. The inverse z-transform can be calculated by means of such tables. Explicit calculation is significantly more difficult, since it involves computation of a contour integral (see Appendix).

5. If the main objective is to determine the system output y_k for a specific input u_k, it may well be easier to perform the convolution of the input with the impulse response of the system, or to iterate the difference equation. The main purpose of z-transforms in this context is to predict the system response using the poles and zeros of the pulse transfer function, as it will be discussed in Chapter 4.

$\mathcal{F}(s)$	$f(kT)$	$F(z)$
—	$1,\ k=0;\ 0,\ k\neq 0$	1
—	$1,\ k=m;\ 0,\ k\neq m$	z^{-m}
$\frac{1}{s}$	$1(kT)$	$\frac{z}{z-1}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s^3}$	$\frac{1}{2!}(kT)^2$	$\frac{T^2}{2}\frac{z(z+1)}{(z-1)^3}$
$\frac{1}{s^4}$	$\frac{1}{3!}(kT)^3$	$\frac{T^3}{6}\frac{z(z^2+4z+1)}{(z-1)^4}$
$\frac{1}{s^m}$	$\lim_{a\to 0}\frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}}{\partial a^{m-1}}e^{-akT}$	$\lim_{a\to 0}\frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}}{\partial a^{m-1}}\frac{z}{z-e^{-aT}}$
$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$\frac{1}{(s+a)^3}$	$\frac{1}{2}(kT)^2e^{-akT}$	$\frac{T^2}{2}e^{-aT}\frac{z(z+e^{-aT})}{(z-e^{-aT})^3}$
$\frac{1}{(s+a)^m}$	$\frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}}{\partial a^{m-1}}e^{-akT}$	$\frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}}{\partial a^{m-1}}\frac{z}{z-e^{-aT}}$
$\frac{a}{s(s+a)}$	$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$

$\mathcal{F}(s)$	$f(kT)$	$F(z)$
$\dfrac{a}{s^2(s+a)}$	$\dfrac{1}{a}(akT-1+e^{-akT})$	$\dfrac{z\big[(aT-1+e^{-aT})z+(1-e^{-aT}-aTe^{-aT})\big]}{a(z-1)^2(z-e^{-aT})}$
$\dfrac{b-1}{(s+a)(s+b)}$	$e^{-akT}-e^{-bkT}$	$\dfrac{(e^{-aT}-e^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
$\dfrac{s}{(s+a)^2}$	$(1-akT)e^{-akT}$	$\dfrac{z\big[z-e^{-aT}(1+aT)\big]}{(z-e^{-aT})^2}$
$\dfrac{a^2}{s(s+a)^2}$	$1-e^{-akT}(1+akT)$	$\dfrac{z\big[z(1-e^{-aT}-aTe^{-aT})+e^{-2aT}-e^{-aT}+aTe^{-aT}\big]}{(z-1)(z-e^{-aT})^2}$
$\dfrac{(b-a)s}{(s+a)(s+b)}$	$be^{-bkT}-ae^{-akT}$	$\dfrac{z\big[z(b-a)-(be^{-aT}-ae^{-bT})\big]}{(z-e^{-aT})(z-e^{-bT})}$
$\dfrac{a}{s^2+a^2}$	$\sin akT$	$\dfrac{z\sin aT}{z^2-(2\cos aT)z+1}$
$\dfrac{s}{s^2+a^2}$	$\cos akT$	$\dfrac{z(z-\cos aT)}{z^2-(2\cos aT)z+1}$
$\dfrac{s+a}{(s+a)^2+b^2}$	$e^{-akT}\cos bkT$	$\dfrac{z(z-e^{-aT}\cos bT)}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$
$\dfrac{b}{(s+a)^2+b^2}$	$e^{-akT}\sin bkT$	$\dfrac{ze^{-aT}\sin bT}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$
$\dfrac{a^2+b^2}{s\big[(s+a)^2+b^2\big]}$	$1-e^{-akT}(\cos bkT + \dfrac{a}{b}\sin bkT)$	$\dfrac{z(Az+B)}{(z-1)\big(z^2-2e^{-aT}(\cos bT)z+e^{-2aT}\big)}$ $A=1-e^{-aT}\cos bT-\dfrac{a}{b}e^{-aT}\sin bT$ $B=e^{-2aT}+\dfrac{a}{b}e^{-aT}\sin bT-e^{-aT}\cos bT$

3.5 Summary

The main learning outcomes of this chapter can be summarized as follows:

- The pulse transfer function:

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}_{t=kT}\right\},$$

  gives the sampled output response of a continuous system $G(s)$, when the input is supplied by a ZOH.

- The ideal ZOH transfer function $ZOH(s)$ has

  gain: $\big|ZOH(j\omega)\big| = T\big|\mathrm{sinc}(\omega T)\big|$, phase: $\arg\big\{ZOH(j\omega)\big\} = -\omega T/2$.

- The exact response y_k of a sampled data system to a discrete input signal u_k can be determined from $G(z)$ by:
  - computing $Y(z) = G(z)U(z)$ and $y_k = \mathcal{Z}^{-1}\{Y(z)\}$;
  - or by iterating the discrete time recurrence equation implied by $G(z)$;
  - or by convolution: $y_k = g_k * u_k$.

4 Step response and pole locations

As for continuous systems, the poles, zeros and the d.c. gain of a discrete transfer function provide all the information needed to estimate the response. When designing a controller, an understanding of how the poles affect the approximate response is often more insightful than the ability to compute the exact response. This chapter provides a characterization of the system response in terms of the locations of the poles of the discrete transfer function.

4.1 Poles in the z-plane

We will start with the z-transform of a cosine which grows or decays exponentially. This will provide insight on the location of poles in the z-plane and their effect on the response of the system.

Let $y(t) = e^{-at}\cos(bt)\mathcal{U}(t)$ (where $\mathcal{U}(t)$ is the unit step function). We then have that

$$\begin{aligned} y(kT) &= r^k \cos(k\theta)\mathcal{U}(kT), && \text{with } r = e^{-aT},\ \theta = bT, \\ &= r^k \tfrac{1}{2}(e^{jk\theta} + e^{-jk\theta})\mathcal{U}(kT). \end{aligned}$$

Its z-transform is given by

$$\begin{aligned} Y(z) &= \tfrac{1}{2}\sum_{k=0}^{\infty} r^k e^{jk\theta} z^{-k} + \tfrac{1}{2}\sum_{k=0}^{\infty} r^k e^{-jk\theta} z^{-k} \\ &= \frac{1}{2}\frac{1}{1-re^{j\theta}z^{-1}} + \frac{1}{2}\frac{1}{1-re^{-j\theta}z^{-1}} && [\text{for } |z| > r] \\ &= \frac{1}{2}\frac{z}{z-re^{j\theta}} + \frac{1}{2}\frac{z}{z-re^{-j\theta}} \\ &= \frac{z(z-r\cos\theta)}{(z-re^{j\theta})(z-re^{-j\theta})}. && \left[= \frac{z(z-r\cos\theta)}{z^2 - 2r(\cos\theta)z + r^2}\right] \end{aligned}$$

Suppose that this signal is generated by applying a unit pulse to the system with transfer function:

$$G(z) = \frac{z(z-r\cos\theta)}{(z-re^{j\theta})(z-re^{-j\theta})}.$$

Figure 14 shows the poles of this transfer function at $z = re^{\pm j\theta}$ and the zeros at $z = 0$ and $z = r\cos\theta$. Figures 15 and 16 give the responses (when the input is a unit pulse) as r and θ vary.

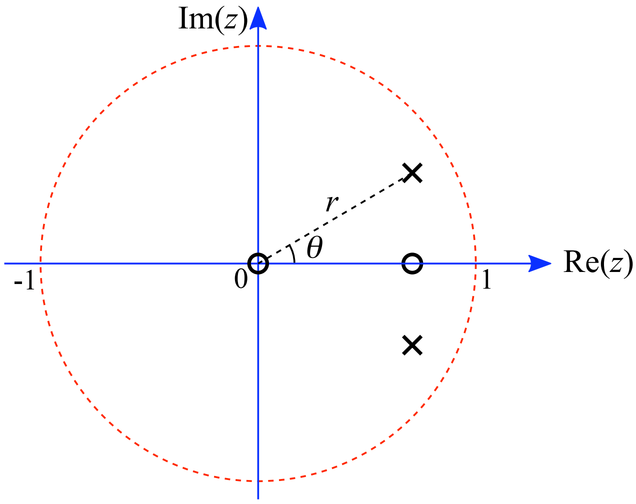


Figure 14: Pole and zero locations, x = pole, o = zero.

As r is varied (Figure 15):

- $r < 1$ gives an exponentially decaying envelope;
- $r = 1$ gives a sinusoidal response with $2\pi/\theta$ samples per period;
- $r > 1$ gives an exponentially increasing envelope.

As θ is varied (Figure 16):

- small values of θ give a frequency much less than the sample rate and lots of samples in a period (less oscillatory response);
- larger θ values give a frequency nearer to the sample rate and fewer samples in a period (more oscillatory response).

Some interesting choices of r and θ are shown below.

- For $\theta = 0$, $Y(z)$ simplifies to:

  $$Y(z) = \frac{z}{z - r},$$

  with one zero at the origin and one pole at $z = r$. This corresponds to an exponentially decaying response (see Example 1).

- For $\theta = 0$ and $r = 1$:

  $$Y(z) = \frac{z}{z - 1},$$

  which is a unit step (see z-transform tables).

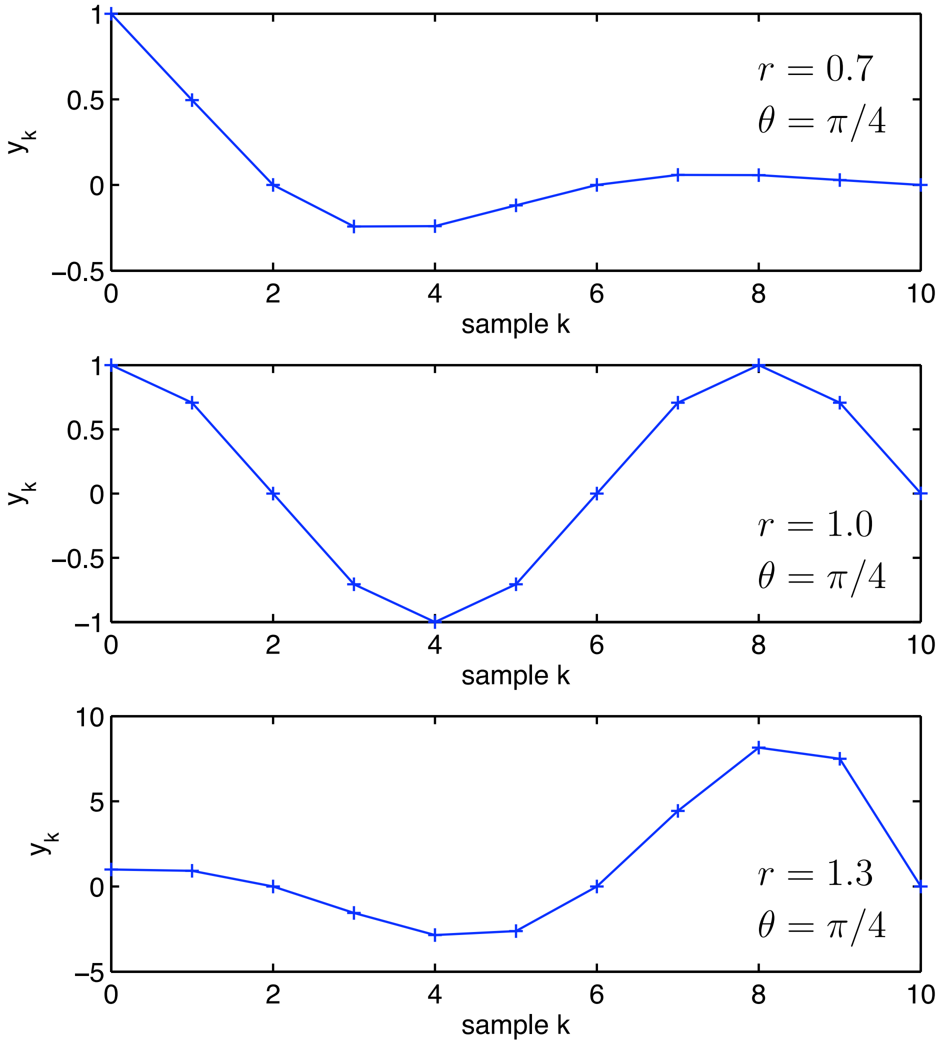


Figure 15: Responses for varying r.

- ▷ For $r = 0$:

$$Y(z) = 1,$$

  which is a unit pulse (see z-transform tables).

- ▷ For $\theta = 0$ and $-1 < r < 0$: We get samples of alternating signs.

For a complete view, Figure 17 summarizes the different responses according to the locations of the poles in the z-plane.

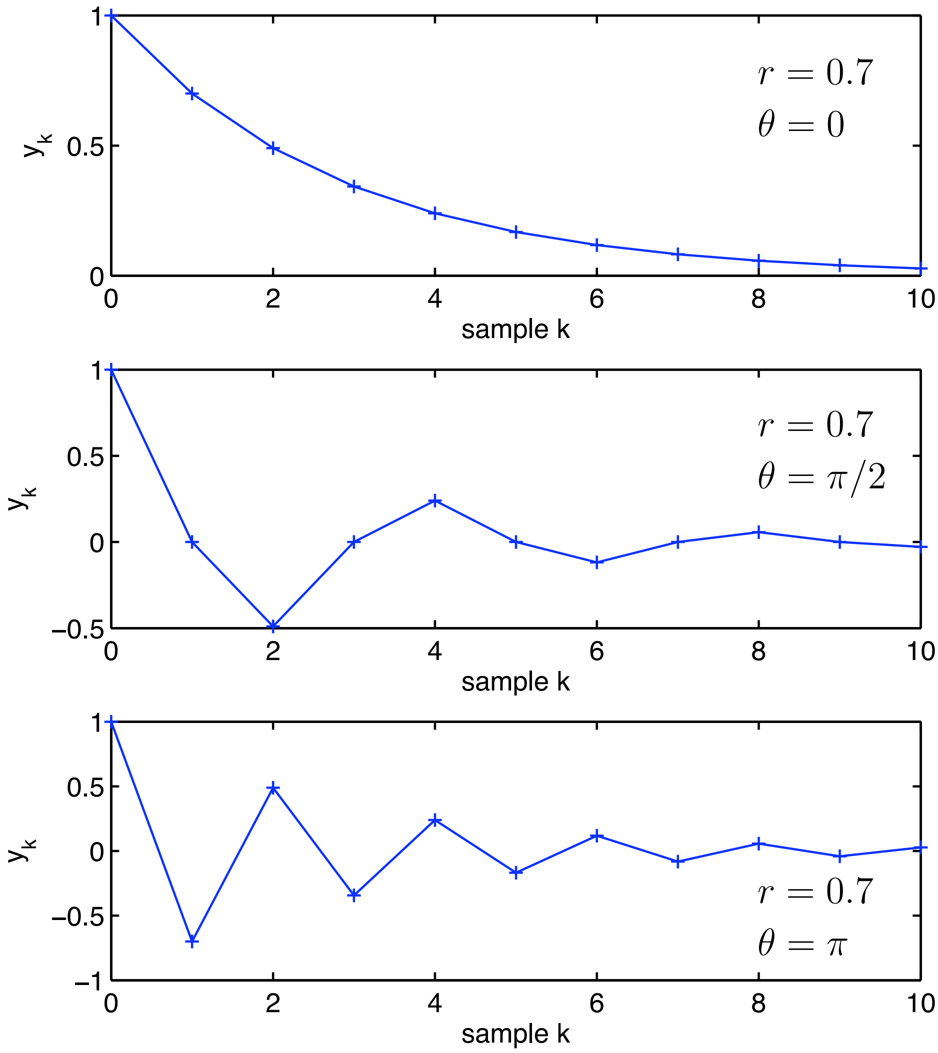


Figure 16: Responses for varying θ.

Implications of pole locations to the response of the system:

- Poles inside the unit circle ($|z| < 1$) $\rightarrow$ stable and convergent response.
- Poles outside the unit circle ($|z| > 1$) $\rightarrow$unstable response.
- Poles on the unit circle ($|z| = 1$) $\rightarrow$ constant response or a sustained oscillation.
- Real poles at $0 < z < 1$ $\rightarrow$ exponential pulse response.
- Real poles near $|z| = 1$ are slow.
- Real poles near $z = 0$ are fast.
- Complex poles give faster responses as their angle to the real axis increases.

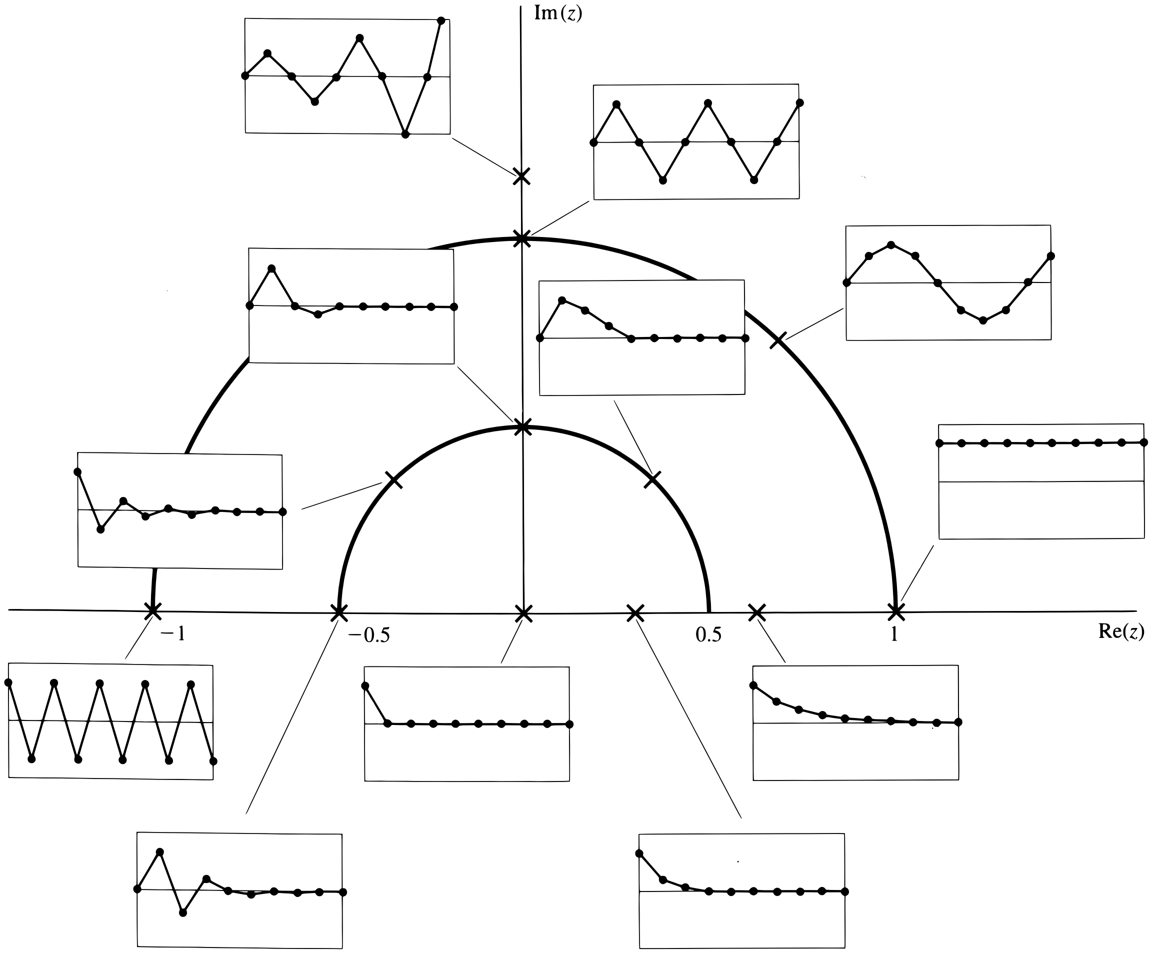


Figure 17: Pulse response for various pole locations (from [Franklin et al, 1998] p.126).

▷ Complex poles give a more resonant response as they get nearer the unit circle.

These observations are in direct comparison with poles in the s-plane. Moreover, notice that stability here is defined informally; we discuss about stability more formally in the next chapter.

4.2 The mapping between s-plane and z-plane

It turns out that the pole structure in the example of the previous section holds true more generally. To this end, consider the z-transforms in pages 31 and 32. It can be noticed, that if a continuous-time signal $F(s)$ has a pole at $s = a$, then the z-transform $F(z)$ always has a pole at $z = e^{aT}$. This is to be expected since e^{-sT} and z^{-1} encode a delay in the continuous and discrete setting, respectively, but we formalize it below.

Fact 3. *If s is a pole of a continuous time transfer function $F(s)$, then*

$$z = e^{sT},$$

is a pole of the discrete time transfer function $F(z)$. Note that this relationship does not hold for the zeros of $F(s)$ and $F(z)$.

Therefore, the pole locations of a continuous signal in the s-plane map to the pole locations of the sampled signal in the z-plane under the transformation $z = e^{sT}$. Writing $s = \sigma + j\omega$, we obtain

$$z = e^{(\sigma+j\omega)T} = e^{\sigma T} e^{j\omega T} \quad \Longrightarrow \quad \begin{cases} |z| = e^{\sigma T}, \\ \arg(z) = \omega T. \end{cases}$$

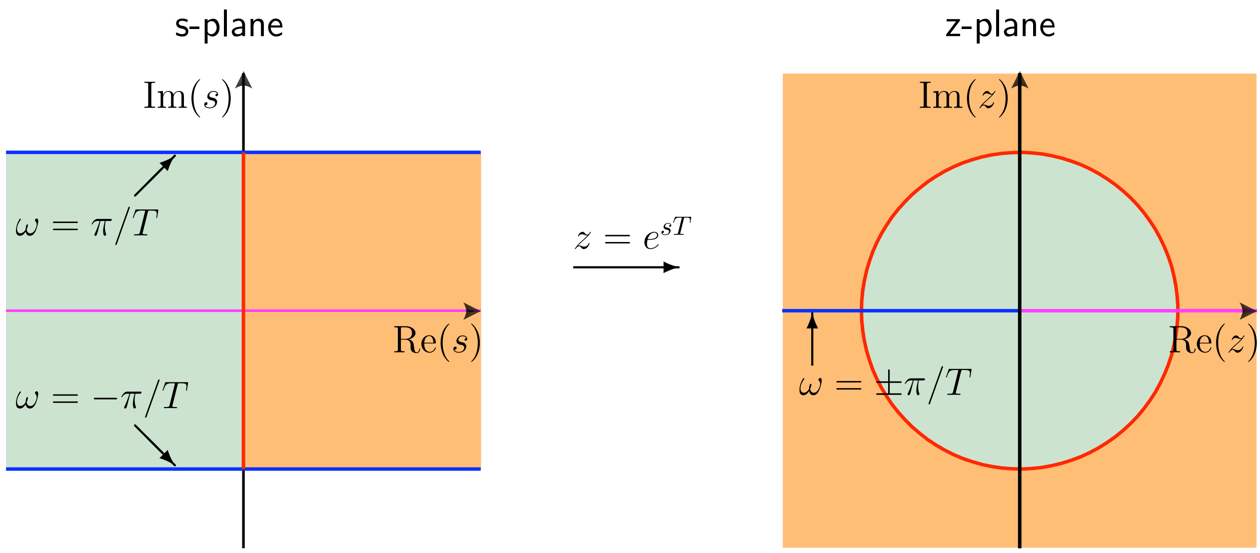


Figure 18: The mapping $z = e^{sT}$ from s-plane to z-plane; the colour coding shows the corresponding regions in the two planes.

This mapping has the following implications:

imaginary axis ($s = j\omega$, $\sigma = 0$)	$\longrightarrow$	unit circle ($z = e^{j\omega T}$);
left-half plane ($\sigma < 0$)	$\longrightarrow$	inside of the unit circle ($\lvert e^{\sigma T}\rvert < 1$);
right-half plane ($\sigma > 0$)	$\longrightarrow$	outside of the unit circle ($\lvert e^{\sigma T}\rvert > 1$);

[i.e., *z-plane poles inside the unit circle are stable*]

negative real axis ($\omega = 0$, $-\infty < \sigma < 0$) $\longrightarrow$ real axis ($0 < z < 1$);
positive real axis ($\omega = 0$, $0 < \sigma < \infty$) $\longrightarrow$ real axis ($z > 1$);
negative infinity ($\sigma \to -\infty$) $\longrightarrow$ origin ($z = 0$).

The mapping from s-plane to z-plane is *not* an one-to-one mapping, because

$$s = s_1 \longrightarrow z = e^{s_1 T} \quad \text{implies} \quad s = s_1 + j2n\pi/T \longrightarrow z = e^{s_1 T} e^{j2n\pi} = e^{s_1 T},$$

for any integer n. On the basis of Nyquist's sampling theorem, which states that we need to sample at a frequency at least twice the signal bandwidth, i.e., $\frac{1}{T} > 2\frac{|\omega|}{2\pi}$, we only consider the part of the s-plane within the half-sampling frequency, $|\omega| < \pi/T$ which maps one-to-one onto the entire z-plane (Figure 18).

4.3 Damping ratio and natural frequency

Recall the s-plane poles with natural frequency ω_0, damping ratio $\zeta < 1$:

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0 \quad \Longrightarrow \quad s = -\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2}$$

The z-plane poles under the mapping $z = e^{sT}$ are

$$z = re^{j\theta} \quad \text{with} \quad r = e^{-\zeta\omega_0 T}, \quad \theta = \omega_0 T\sqrt{1-\zeta^2}.$$

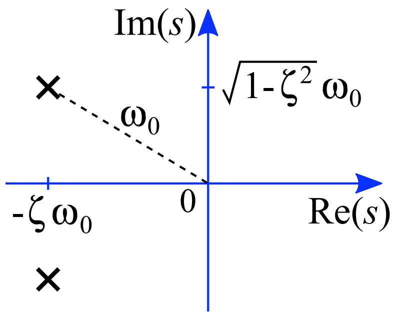


Figure 19: s-plane poles.

We thus have that:

- Lines of fixed damping ζ, which radiate from the origin of the s-plane map to logarithmic spirals in the z-plane, as shown in Figure 20.
- Lines of constant natural frequency, ω_0 which are circles in the s-plane, map to the shapes shown in Figure 20.

4.4 System specifications*

Control systems may be designed to satisfy certain specifications. However, the response of the controlled system is often dictated only by the dominant (i.e., the slowest) poles. Therefore, we aim for a closed-loop response dominated by a pair

*[Franklin et al, 1998] §2.1.6, 2.1.7, 2.2.2, 2.2.4

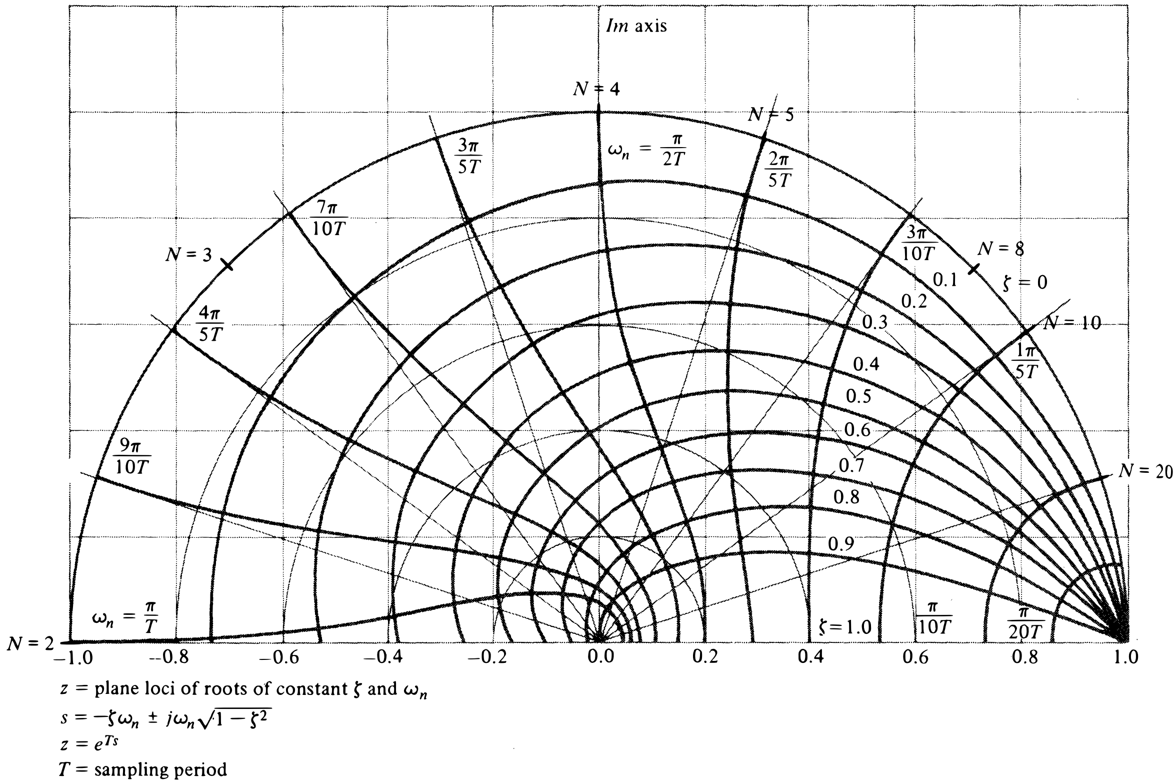


Figure 20: z-plane lines of constant damping ratio and natural frequency (from [Franklin et al, 1998] p.125). Note that ω_n in the figure corresponds to ω_0 in these notes.

of complex conjugate poles constituting an approximation to an underdamped 2nd-order system, and impose performance specifications for the resulting system.

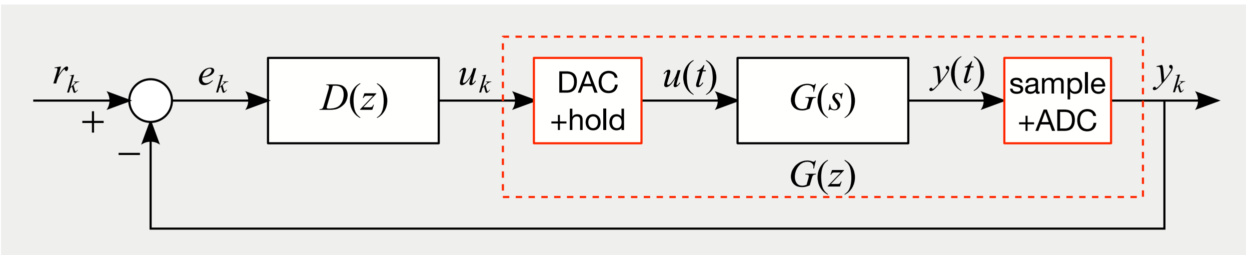


Figure 21: Discrete controller and continuous plant.

The discrete control system in Figure 21 has a closed-loop transfer function

$$\frac{Y(z)}{R(z)} = \frac{D(z)G(z)}{1 + D(z)G(z)},$$

where $Y(z)$ and $R(z)$ are the z-transforms of the output and reference signal, respectively. Approximating its response by that of a system with a pair of complex conjugate poles, we need to specify the locations of the dominant z-plane poles, i.e., the roots of $1 + D(z)G(z) = 0$.

Typically we aim for a damping factor ζ around 0.7 and natural frequency ω_0 as high

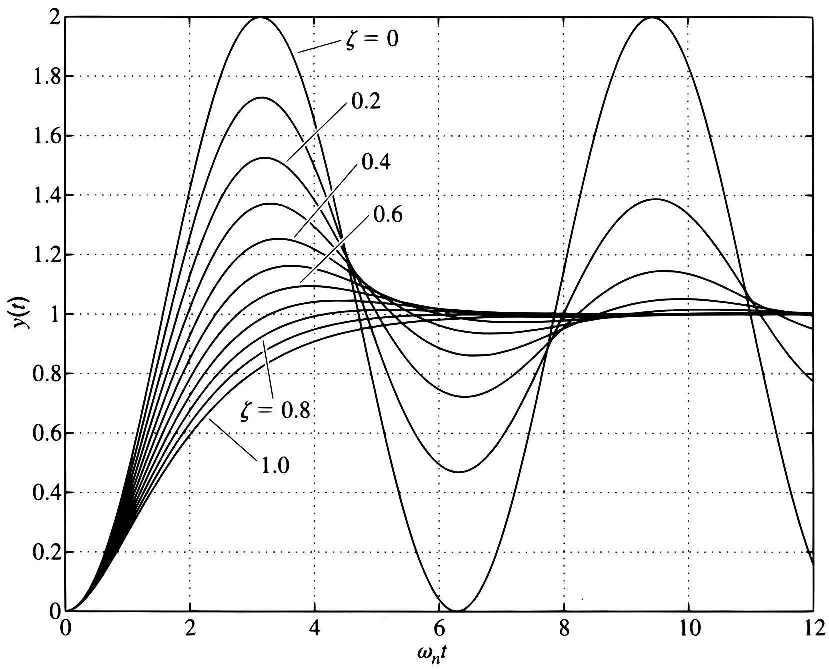


Figure 22: Second order system step responses (from [Franklin et al, 1998] p.19, see also HLT p.168).

as reasonably possible to make the response as fast as possible (see also Figure 27 to justify this choice). Examples of second order system step responses and design specifications are shown in Figures 22 and 23, respectively. Actual design criteria include:

Rise time:	t_r;
Settling time:	t_s;
Peak overshoot:	M_p;
Steady state error:	e_{ss};
Gain and Phase margin specifications.	

For a 2nd-order underdamped system, we can obtain some approximate relationships between these criteria and the pole positions:

Rise time (10% to 90%):	$t_r \approx 1.8/\omega_0$;	
Settling time (to 1%):	$t_s \approx 4.6/(\zeta\omega_0)$;	
Peak overshoot:	$M_p \approx e^{-\pi\zeta/\sqrt{1-\zeta^2}}$,	(for $0 \le \zeta \le 1$);
Steady state error:	$e_{ss} = \lim_{z\to 1}(z-1)E(z)$;	
Phase margin:	$\phi_{PM} \approx 100\zeta$, $\phi_{PM} \le 60^\circ$	(ϕ_{PM} in degrees).

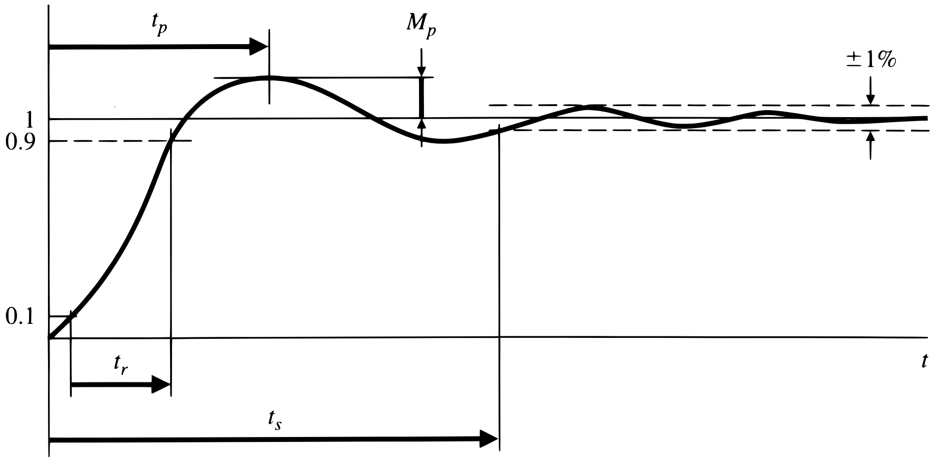


Figure 23: Step response design criteria (from [Franklin et al, 1998] p.20).

To design a controller so as to meet a given specification in terms of e_{ss}, t_s, M_p etc., a number of design techniques (including root locus and frequency response methods) exist for placing the z-plane poles at desired locations. However these are beyond the scope of this course. However, we could verify whether certain specifications are met; this is illustrated by means of the following example.

Example 7. *A continuous system with transfer function $G(s) = 1/(s(10s + 1))$ is controlled by a discrete control system with a ZOH, as shown in Figure 21. The closed loop system is required to have: a step response $y(kT)$ with overshoot $M_p < 16\%$ and settling time $t_s < 10$ s; and steady-state error to a unit ramp $e_{ss} < 1$.*
Validate the satisfaction of these specifications if the sample interval is $T = 1$ s and controller is

$$u_k = -0.5u_{k-1} + 13(e_k - 0.88e_{k-1}).$$

1. *Determine the pulse transfer function of $G(s)$ plus the ZOH:*

$$G(z) = (1 - z^{-1})\mathcal{Z}\Big\{\frac{G(s)}{s}\Big\}$$
$$= \frac{(z-1)}{z}\mathcal{Z}\Big\{\frac{0.1}{s^2(s+0.1)}\Big\}.$$

From the z-transform tables with $T = 1$ and $a = 0.1$ we obtain:

$$
\begin{aligned}
G(z) &= \frac{(z-1)}{z}\,\frac{z\big((0.1-1+e^{-0.1})z+(1-e^{-0.1}-0.1e^{-0.1})\big)}{0.1(z-1)^2(z-e^{-0.1})} \\
&= \frac{(0.1-1+e^{-0.1})z+(1-e^{-0.1}-0.1e^{-0.1})}{0.1(z-1)(z-e^{-0.1})} \\
&= \frac{0.0484(z+0.9672)}{(z-1)(z-0.9048)}.
\end{aligned}
$$

2. *Determine the controller transfer function $D(z)$: Using the linearity and delay properties of the z-transform, we obtain*

$$D(z) = \frac{U(z)}{E(z)} = 13\frac{(1-0.88z^{-1})}{(1+0.5z^{-1})} = 13\frac{(z-0.88)}{(z+0.5)}.$$

3. *Verify steady state error specification. The transfer function from r to e is given by*

$$\frac{E(z)}{R(z)} = \frac{1}{1+D(z)G(z)}.$$

and the steady state error is $e_{ss} = \lim_{k\to\infty} e_k$. From the z-transform of a ramp (see z-transform tables) we obtain that

$$R(z) = \frac{Tz}{(z-1)^2}.$$

By the final value theorem property of the z-transform we thus have that

$$
\begin{aligned}
e_{ss} &= \lim_{z\to 1}(z-1)E(z) = \lim_{z\to 1}\Big\{(z-1)\frac{Tz}{(z-1)^2}\frac{1}{1+D(z)G(z)}\Big\} \\
&= \lim_{z\to 1}\frac{Tz}{(z-1)(1+D(z)G(z))}, \quad \textit{[poles of } (z-1)E(z) \textit{ in unit circle]} \\
&= \frac{1-0.9048}{0.0484(1+0.9672)D(1)} \\
&\approx \frac{1}{D(1)} = \frac{(1+0.5)}{13(1-0.88)} = 0.96.
\end{aligned}
$$

Therefore, $e_{ss} < 1$ as required.

4. *Translating the specification on overshoot and settling time into constraints on the poles of the transfer function from r to y, we obtain:*

overshoot: $M_p < 16\% \quad \Longrightarrow \quad \zeta > 0.5;$

settling time: $t_s < 10 \quad \Longrightarrow \quad \zeta\omega_0 > 0.46 \Rightarrow$ *radius of poles* $< 0.63.$

The transfer function from r to y is given by

$$\frac{Y(z)}{R(z)} = \frac{D(z)G(z)}{1 + D(z)G(z)}.$$

It has the following two dominant poles:

$$z = -0.050 \pm j0.304 = re^{\pm j\theta} \quad \begin{cases} r = |-0.050 + j0.304| = 0.31; \\ \theta = \tan^{-1}(0.304/0.05) = 1.73. \end{cases}$$

Notice that, the roots of $1 + D(z)G(z) = 0$, *are* $z = 0.876,\ -0.050 \pm j0.304$. *However the transfer function* $\dfrac{Y(z)}{R(z)}$ *has a zero at* $z = 0.88$ *which effectively cancels the pole at* $z = 0.876$.

By the dominant poles are we have that the specification $r < 0.63$ *is satisfied. Finally, to check that* $\zeta > 0.5$, *by p. 40 we have that*

$$\left.\begin{aligned} r &= e^{-\zeta\omega_0 T} \\ \theta &= \omega_0 T\sqrt{1-\zeta^2} \end{aligned}\right\} \implies \frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{\ln(1/r)}{\theta} = 0.680 \implies \zeta = 0.56.$$

Hence, the controller meets the specifications, as shown by the closed loop step response in Figure 24.

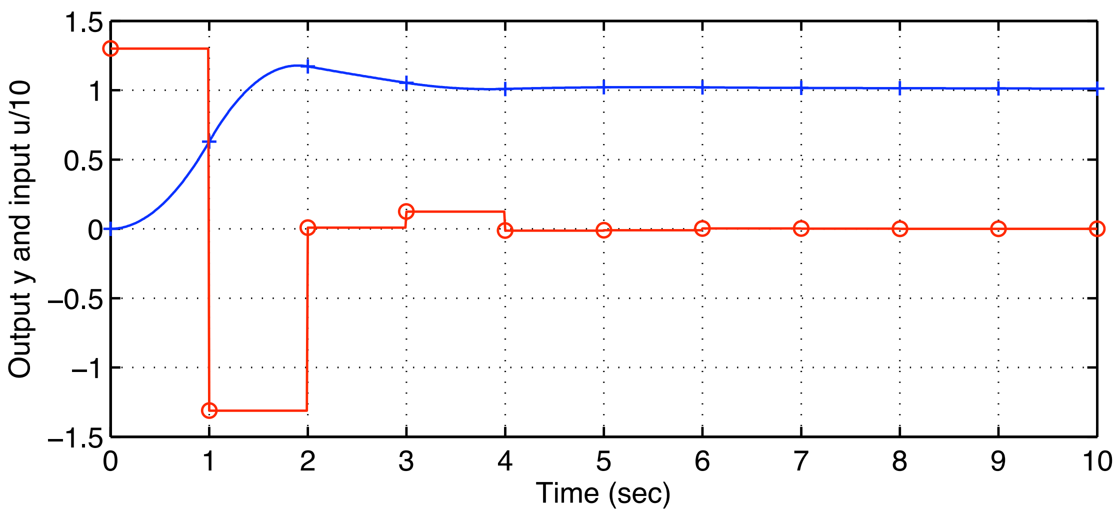


Figure 24: Step response, −+− = output y, −o− = input $u \times 0.1$.

4.5 Summary

The main learning outcomes of this chapter can be summarized as follows:

- The pulse response of a 2nd order system is investigated as the radius and argument of its z-plane poles are varied.
- For a sampled data system with a ZOH:

  if $s = p_i$ is a pole of $G(s)$, then $z = e^{p_i T}$ is a pole of $G(z)$.
- The locus of $s = \sigma + j\omega$ under the mapping $z = e^{sT}$ is considered:
  - ▷ the left half plane ($\sigma < 0$) maps to the unit disk ($|z| < 1$);
  - ▷ s-plane poles with damping ratio ζ, natural frequency ω_0 map to z-plane poles with $|z| = e^{-\zeta\omega_0 T}$, $\arg(z) = \sqrt{1-\zeta^2}\omega_0 T$.
- Design specifications (rise time, settling time, overshoot) translate into constraints on the ζ and ω_0 values of the dominant closed loop poles, that can be verified for the designed controller.

5 Discrete time linear systems in state space form

We have already seen that discrete time systems emanate from continuous ones via sampling. In particular, in the previous chapters we focused on such systems that are linear, analyzed them and discussed their stability properties by means of transfer functions. Here, we will revisit the problem of analyzing discrete time linear systems, but we will do so directly in the discrete time domain.

To this end, this chapter will address the following problems:

1. Provide a unified modelling formalism to represent discrete time linear systems, referred to as *state space* form.
2. We will provide an explicit characterization of solutions to discrete time linear systems and discuss how these can be computed.
3. We will discuss how stability can be inferred directly from the system's state space representation, without resorting to poles and transfer functions.

5.1 State space representation

Consider a generalization of the plant that is shown in Figure 2, that may have multiple inputs, encoded by the vector $u(t) \in \mathbb{R}^m$, and multiple outputs, encoded by the vector $y(t) \in \mathbb{R}^p$. Rather than describing the plant by means of a transfer function, we have already seen in earlier lectures of A2 Introduction to Control Theory that, if it is given by a continuous time linear and time-invariant system with an input vector $u(t) \in \mathbb{R}^m$ and an output vector $y(t) \in \mathbb{R}^p$, then we can equivalently represent it in the so called state space form. Specifically, the state space form of a given plant is given by

$$\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t),$$
$$y(t) = \bar{C}x(t) + \bar{D}u(t).$$

Vector $x(t) \in \mathbb{R}^n$ denotes the so called system state that acts as an "internal" vector, whose elements correspond to physical quantities that change over time, hence their evolution is described by means of ordinary differential equations (ODEs).

Stacking these ODEs together gives rise to the first of the equations in the state space description above. The second equation is an algebraic one and encodes the information that is available in the output of the system by means of sensor measurements. These equations together constitute the state space form of a continuous, time-invariant, linear system and are fully parameterized by the matrices $\bar{A} \in \mathbb{R}^{n\times n}$, $\bar{B} \in \mathbb{R}^{n\times m}$, $\bar{C} \in \mathbb{R}^{p\times n}$ and $\bar{D} \in \mathbb{R}^{p\times m}$.

With reference to Figure 2, we would like to analyze the sampled data system with input u_k and output y_k, where $k \geq 0$ corresponds to any given sampling instance. Recall that for any $t \in [kT, (k+1)T]$ the state solution of the continuous time linear system can be written as

$$x(t) = e^{\bar{A}(t-kT)}x_k + \int_{kT}^{t} e^{\bar{A}(t-\tau)}\bar{B}u(\tau)d\tau,$$

where $x_k = x(kT)$ plays the role of the initial condition of the state over the sampling interval under consideration. As the initial condition is not $x(0)$, the term kT appears in the exponential of the first term and the lower limit of the integral in the second term, to account for the fact that the initial time of the state evolution is no longer zero but kT. Notice that between consecutive sampling instances kT and $(k+1)T$, the control input remains constant to u_k as an effect of the ZOH, i.e., we have that

$$u(t) = u_k \quad \text{for all} \quad t \in [kT, (k+1)T).$$

Evaluating the state solution at $t = (k+1)T$ and setting $x_{k+1} = x((k+1)T)$, we have that

$$\begin{aligned} x_{k+1} &= e^{\bar{A}T}x_k + \left(\int_{kT}^{(k+1)T} e^{\bar{A}((k+1)T-\tau)}\bar{B}d\tau\right)u_k \\ & \quad [\text{change of variables:} \quad \tau \leftarrow \tau - kT] \\ &= e^{\bar{A}T}x_k + \left(\int_{0}^{T} e^{\bar{A}(T-\tau)}\bar{B}d\tau\right)u_k. \end{aligned}$$

Moreover, the output equation at the sampled time instances is given by

$$y_k = \bar{C}x_k + \bar{D}u_k.$$

where $y_k = y(kT)$. Combining the derived equations for the (next) state x_{k+1} and the output y_k as functions of x_k and u_k we obtain the state space description of a sampled data, discrete time linear system. This is summarized below.

A sampled data, discrete time linear system is said to be in state space form if it can be written as

$$x_{k+1} = Ax_k + Bu_k,$$
$$y_k = Cx_k + Du_k,$$

where $A = e^{\bar{A}T} \in \mathbb{R}^{n\times n}$, $B = \int_0^T e^{\bar{A}(T-\tau)}\bar{B}d\tau \in \mathbb{R}^{n\times m}$, $C = \bar{C} \in \mathbb{R}^{p\times n}$ and $D = \bar{D} \in \mathbb{R}^{p\times m}$. Matrices $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ correspond to the state space form of the continuous time, linear, time-invariant system prior to sampling, and T denotes the sampling period.

Example 8. *Consider a plant given by the continuous time linear system*

$$\dot{x}(t) = -x(t) + u(t),$$
$$y(t) = x(t).$$

The system is already in state space form with $\bar{A} = -1$, $\bar{B} = 1$, $\bar{C} = 1$ and $\bar{D} = 0$ (all scalars). To compute the state space description of the associated sampled data system with sampling period T, we need to determine matrices (A, B, C, D). To this end, notice that $C = \bar{C} = 1$ and $D = \bar{D} = 0$, while

$$A = e^{\bar{A}T} = e^{-T},$$
$$B = \int_0^T e^{\bar{A}(T-\tau)}\bar{B}d\tau = \int_0^T e^{\tau-T}d\tau = 1 - e^{-T}.$$

Therefore, the associated discrete time system is given by

$$x_{k+1} = e^{-T}x_k + (1 - e^{-T})u_k,$$
$$y_k = x_k.$$

For the case where unlike the preceding example the system is non-scalar, given a continuous time state space description we can construct the discrete time one by means of the following two steps:

1. **Step 1: Compute the matrix exponential $e^{\bar{A}T}$.** This can be computed as discussed in earlier lectures of A2 Introduction to Control Theory. In particular, if $\bar{A}$ is diagonalizable, we can compute the matrix exponential as

$e^{\bar{A}T} = We^{\Lambda T}W^{-1}$, where W is a matrix with the eigenvectors of $\bar{A}$ as its columns (invertible as for diagonalizable matrices eigenvectors are linearly independent) and Λ is a diagonal matrix with the eigenvalues of $\bar{A}$ as the diagonal entries. Since Λ is diagonal, computing $e^{\Lambda T}$ becomes then easier as it involves computing scalar exponentials corresponding to the entries of Λ, i.e.,

$$e^{\Lambda T} = \begin{bmatrix} e^{\lambda_1 T} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 T} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n T} \end{bmatrix}$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\bar{A}$.

2. **Step 2: Compute the integral $\int_0^T e^{\bar{A}(T-\tau)}\bar{B}d\tau$.** This should be performed elementwise for every entry of the matrix $e^{\bar{A}(T-\tau)}\bar{B}$ that appears in the integrand, which in turn involves the matrix exponential.

5.2 Solutions to discrete time linear systems

Consider a discrete time linear system in state space form, and an arbitrary initial state x_0. For a given sequence of inputs u_k, $k = 1, \dots,$ we can provide an explicit characterization of the solution to such systems. In particular, the state x_k and output y_k solution for any instance k, are given by

$$x_k = \overbrace{A^k x_0}^{\text{zero input transition}} + \overbrace{\sum_{i=0}^{k-1} A^{k-i-1} B u_i}^{\text{zero state transition}}$$

$$y_k = \underbrace{CA^k x_0}_{\text{zero input response}} + \underbrace{\sum_{i=0}^{k-1} CA^{k-i-1} B u_i + D u_k}_{\text{zero state response}}.$$

We will not prove that the expressions for x_k and y_k satisfy the difference and output equation, respectively; this can be achieved using induction. Notice that once x_k is computed, the output solution can be directly determined by means of $y_k = Cx_k + Du_k$. The first term in the state solution is the so called *zero input transition*, i.e., the solution of the system if it was autonomous, or in other words if $u_i = 0$ for all i. The second term is the so called *zero state transition*, and it is a discrete time convolution corresponding to the solution of the system when the initial state is zero. The interpretation of the *zero input response* and the *zero state response* is analogous, referring to the output y_k instead.

These solutions are in direct correspondence with the ones for continuous time linear systems; the only difference is that the convolution integral is replaced with a summation involved in discrete time convolution, while the matrix exponential is replaced by A^k. The latter is the term that is the most difficult to compute so that we determine the state output solutions x_k and y_k. We discuss its computation separately for systems with a diagonalizable and a non-diagonalizable matrix A.

5.2.1 Diagonalizable matrices

A matrix is called diagonalizable if its eigenvectors are linearly independent. We have the following sufficient conditions for a matrix to be diagonalizable:

1. If a matrix $A \in \mathbb{R}^{n\times n}$ has distinct eigenvalues (i.e., $\lambda_i \neq \lambda_j$ for all $i \neq j$), then its eigenvectors are linearly independent.
2. If a matrix is symmetric ($A = A^\top \in \mathbb{R}^{n\times n}$), then i) its eigenvalues are real; ii) its eigenvectors are orthonormal (hence linearly independent).

Diagonalizability is a desirable property as, if a matrix A is diagonalizable, it can be decomposed as

$$A = W\Lambda W^{-1},$$

where W is a matrix whose columns are the eigenvectors of A (invertible since the eigenvectors are linearly independent), and Λ is a diagonal matrix whose diagonal entries correspond to the eigenvalues of A. For discrete time linear systems with a diagonalizable matrix A the quantity A^k that appears in the expressions for the state and output solutions can be computed efficiently in closed form.

Fact 4. *Consider a discrete time linear system with matrix $A \in \mathbb{R}^{n\times n}$ being diagonalizable, thus admitting a decomposition $A = W\Lambda W^{-1}$. We then have that*

$$A^k = W\Lambda^k W^{-1},$$

where $\Lambda^k = \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix}$, *with λ_i being the i-th eigenvalue of A.*

Proof: *We show this by means of induction:*

1. *Base case ($k = 0$): We have that $W\Lambda^0 W^{-1} = WW^{-1} = I = A^0$, hence the base case is trivially satisfied.*
2. *Induction hypothesis: Assume the statement holds true for an arbitrary k, i.e., $A^k = W\Lambda^k W^{-1}$.*
3. *Show the claim for the $(k+1)$-th case: By the induction hypothesis we have that $A^k = W\Lambda^k W^{-1}$. Hence,*

$$A^{k+1} = A^k A = W\Lambda^k \overset{I}{\cancel{W^{-1}W}} \Lambda W^{-1} = W\Lambda^{k+1}W^{-1}.$$

Example 9. *Consider a discrete time linear system with difference equation*

$$x_{k+1} = Ax_k = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} x_k.$$

Compute the zero input transition if the initial state is $x_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Step 1: *Compute the eigenvalues of A. Since the matrix is triangular, the eigenvalues correspond to the diagonal elements, i.e., $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = 1$. Since the eigenvalues are distinct we infer that A is diagonalizable.*

Step 2: *Compute the eigenvectors of A. By direct calculation we obtain that*

$$\textit{corresponding to } \lambda_1: \; w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \textit{corresponding to } \lambda_2: \; w_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Step 3: *Since A is diagonalizable, we can write it as $A = W\Lambda W^{-1}$, where*

$$\Lambda = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, \; W = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \;\; \text{and} \;\; W^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

We then have that the zero state transition is given by

$$\begin{aligned} x_k &= A^k x_0 = W\Lambda^k W^{-1} x_0 \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2^k} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 - \frac{1}{2^{k-1}} \\ 1 \end{bmatrix}. \end{aligned}$$

Notice that as $k \to \infty$ the state tends to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

5.2.2 Non-diagonalizable matrices

A matrix $A \in \mathbb{R}^{n\times n}$ is called non-diagonalizable if the number of linearly independent eigenvectors is strictly lower than n. Moreover, not all of its eigenvalues are distinct, and at least one of them would be repeated. If the matrix A of a given discrete time linear system is non-diagonalizable, then we can no longer decompose A using the eigenvector matrix W as this will no longer be invertible (recall that the columns of W are the eigenvectors).

Computing A^k for non-diagonalizable matrices with generic structure is a difficult task; it requires resorting to the so called Jordan form and will not be addressed in these notes. However, for a particular class of non-diagonalizable matrices computing A^k can be performed efficiently. We illustrate this by means of an example.

Example 10. *Consider the matrix*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The diagonal elements of A correspond to its eigenvalues, hence A has a repeated eigenvalue $\lambda = 0$ with multiplicity equal to 3. To decide whether A is diagonalizable we have to compute its eigenvectors. Using $\lambda = 0$ we obtain only one eigenvector, i.e.,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = 0 \;\Rightarrow\; w_2 = w_3 = 0, \quad \textit{eigenvector:} \quad w = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Since we only have $1 < 3$ linearly independent eigenvectors, A is non-diagonalizable.
However, notice that we can still compute A^k efficiently, as $A^3 = 0$ and as a result, $A^k = 0$ for all $k \geq 3$.

Matrices that exhibit this property (excluding the trivial case of a zero matrix), i.e., there exists some integer $\bar{k}$ such that $A^k = 0$ for all $k \geq \bar{k}$, are called Nilpotent. Triangular $n \times n$ matrices with zeros along the main diagonal are all Nilpotent with $\bar{k} \leq n$. For Nilpotent matrices the state solution satisfies

$$x_k = A^k x_0 = 0 \text{ for all } k \geq \bar{k}.$$

In other words, the state of the system becomes zero in finite time ($\bar{k}$ time instances) and stays there from then on. Therefore, for non-diagonalizable but Nilpotent matrices A the system's state solution can be easily computed. Despite the analogies between continuous and discrete time linear systems, this behaviour is not encountered in continuous time systems.

5.3 Stability of discrete time linear systems

In this section we will investigate the stability properties of discrete time linear systems with no inputs, i.e., systems governed by

$$x_{k+1} = Ax_k.$$

In other words, we will analyze the so called zero input transition of the state solution, namely $x_k = A^k x_0$. However, stability is related to the limiting behaviour of the system state x_k as $k \to \infty$. In particular, we will investigate the stability properties of the equilibrium point $x_k = 0$, as if the state becomes zero for some k, we would have that all subsequent states would be zero as well since $x_{k+1} = Ax_k = 0$. We consider the following notions of stability:

1. *Stability:* A system is called stable* if we can stay arbitrarily close enough to 0 if we start sufficiently close to it.

2. *Asymptotic stability:* A system is called asymptotically stable if it is stable, and approaches 0 as time tends to infinity, i.e., $\lim_{k\to\infty} \|x_k\| = 0$. In other words, not only we stay close to 0, but also converge to it.

We then say that a system that is not stable is unstable. We will investigate these notions of stability for systems of the form $x_{k+1} = Ax_k$, separately for diagonalizable and non-diagonalizable A matrices.

5.3.1 Diagonalizable matrices

In the case where A is diagonalizable, following the procedure outlined in Fact 4, we can represent the state solution of the system (i.e., the zero input transition in the absence of inputs) for any k as

$$x_k = A^k x_0 = W\Lambda^k W^{-1} x_0.$$

It follows then that A^k (and as result x_k) will be a linear combination of terms λ_i^k, $i = 1, \ldots, n$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. The coefficients of this linear combination would then depend on the eigenvectors that appear in W and the initial state, and would dictate how the terms λ_i^k, $i = 1, \ldots, n$, mix.

It follows then that the eigenvalues affect the limiting behaviour of the state. Eigenvalues are in general complex, hence we can represent them as $\lambda_i = \sigma_i + j\omega_i$, which implies that $|\lambda_i| = \sqrt{\sigma_i^2 + \omega_i^2}$. It can be then observed that

- If $|\lambda_i| = 1$ then $|\lambda_i|^k = 1$ for all k.

*Formally, a system is called stable if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $\|x_0\| \le \delta$ then $\|x_k\| \le \epsilon$ for all $k \ge 0$.

- If $|\lambda_i| < 1$ then $\lim_{k\to\infty} |\lambda_i|^k = 0$.
- If $|\lambda_i| > 1$ then $\lim_{k\to\infty} |\lambda_i|^k = \infty$.

The relationships above sumarize the limiting behaviour for one of the terms that would appear in the state solution, i.e., the one related to λ_i^k. However, the state solution would be a mix of such terms corresponding to the different eigenvalues of A. We are then able to make the following statements related to stability.

Fact 5 (Stability of $x_{k+1} = Ax_k$ with diagonalizable A)**.** *Consider a discrete time linear system of the form $x_{k+1} = Ax_k$ with $A \in \mathbb{R}^{n\times n}$ diagonalizable. Denote by λ_i, $i = 1, \ldots, n$, the eigenvalues of A. We then have that the system is*

- *Stable (for all initial conditions) if and only if $|\lambda_i| \leq 1$ for all $i = 1, \ldots, n$.*
- *Asymptotically stable (for all initial conditions) if and only if $|\lambda_i| < 1$ for all $i = 1, \ldots, n$.*
- *Unstable (for some initial conditions) if and only if there exists i such that the corresponding eigenvalue has $|\lambda_i| > 1$.*

Note that it suffices that at least one eigenvalue has magnitude greater than one for the system to be unstable. Moreover, if at least one eigenvalue has magnitude equal to one, while all other ones have magnitude strictly lower than one, then the system is stable but not asymptotically stable. The reason is that the contribution of that particular eigenvalue in the state solution is bounded (constant or periodic, hence we only stay "close" to zero but do not converge to it). Notice that this is the case in Example 9 as one of the eigenvalues is equal to one, hence in the limit the state does not tend to zero for the particular choice of the initial condition.

These implications of stability are in direct correspondence with the less formal stability observations according to the poles of the system's transfer function. In particular, poles and eigenvalues convey the same information about stability. In fact, a pole is also an eigenvalue of the system's A matrix; vice verca, if the there are no pole-zero cancellations, an eigenvalue is also a pole of the system's transfer function.

5.3.2 Non-diagonalizable matrices

In the case where A is non-diagonalizable but Nilpotent, then the state reaches zero in finite time and stays there afterwards (see Example 11), hence we have stability and in fact we achieve this in finite time. However, for non-diagonalizable matrices with generic structure, eigenvalues offer less information as far as determining the system's stability is concerned. To see this, recall first that for non-diagonalizable matrices at least one eigenvalue will be repeated. If for all eigenvalues $|\lambda_i| \leq 1$, $i = 1, \ldots, n$, but the repeated eigenvalue is such that $|\lambda_i| = 1$ then, unlike diagonalizable matrices, the response may be stable or unstable according to the initial condition (and how this relates to eigenvectors). We illustrate this by means of an example.

Example 11. *Consider the system*

$$x_{k+1} = Ax_k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k.$$

The diagonal elements of A correspond to its eigenvalues, hence A has a repeated eigenvalue $\lambda = 1$ with multiplicity equal to 2. To decide whether A is diagonalizable we have to compute its eigenvectors. Using $\lambda = 1$ we obtain only one eigenvector, i.e.,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \Rightarrow w_2 = 0, \quad \textit{eigenvector:} \quad w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since we only have $1 < 2$ linearly independent eigenvectors, A is non-diagonalizable.

By computing a few powers and observing the pattern, notice that for all $k \geq 1$

$$A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

For different initial states we observe the following:

$$x_k = A^k x_0 = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$x_k = A^k x_0 = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ 1 \end{bmatrix}.$$

Notice that the first initial state (aligned with the eigenvector) leads to a stable solution, while for the second initial state $\|x_k\|$ tends to inifinity as k increases.

However, looking into the eigenvalues we can still claim the following.

Fact 6 (Stability of $x_{k+1} = Ax_k$ with non-diagonalizable A)**.** *Consider a discrete time linear system of the form $x_{k+1} = Ax_k$ with $A \in \mathbb{R}^{n\times n}$ non-diagonalizable. Denote by λ_i, $i = 1, \ldots, n$, the eigenvalues of A (at least one of them would be repeated). We then have that the system is*

- *Asymptotically stable (for all initial conditions) if and only if $|\lambda_i| < 1$ for all $i = 1, \ldots, n$.*
- *Unstable (for some initial conditions) if there exists i such that the corresponding eigenvalue has $|\lambda_i| > 1$.*

5.4 Summary

The main learning outcomes of this chapter can be summarized as follows:

- Given a continuous time linear system with state space description encoded by $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$, the associated sampled data (with sampling period T), discrete time linear system can be written in state space form as

  $$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Du_k, \end{aligned}$$

  where $A = e^{\bar{A}T} \in \mathbb{R}^{n\times n}$, $B = \int_0^T e^{\bar{A}(T-\tau)}\bar{B}d\tau \in \mathbb{R}^{n\times m}$, $C = \bar{C} \in \mathbb{R}^{p\times n}$ and $D = \bar{D} \in \mathbb{R}^{p\times m}$.

- The state and output solutions to discrete linear systems are, respectively, given by

$$x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} B u_i,$$
$$y_k = C A^k x_0 + \sum_{i=0}^{k-1} C A^{k-i-1} B u_i + D u_k.$$

It was shown that for

- *Diagonalizable A:* $A^k = W \Lambda^k W^{-1}$, where W includes the eigenvectors of A as columns, while Λ is a diagonal matrix with the eigenvalues of A on its diagonal.
- *Non-diagonalizable A:* For the particular class of Nilpotent matrices, $A^{\bar{k}} = 0$ for some integer $\bar{k}$.

- The stability of discrete time systems of the form $x_{k+1} = Ax_k$ was analyzed and conditions to guarantee stability based on the eigenvalues of matrix A were derived. Irrespective of whether A is diagonalizable or not, if $\lambda_1, \ldots, \lambda_n$ denote its eigenvalues, we can claim that:
  - *Asymptotically stable system* if and only if $|\lambda_i| < 1$ for all $i = 1, \ldots, n$.
  - *Unstable system* if there exists i such that $|\lambda_i| > 1$.

6 Appendix: The inverse z-transform

We provide in this appendix a formal definition of the inverse z-transform*. To this end, consider a discrete time signal $f(kT)$, $k = 0, 1, \ldots$, and its z-transform (assuming that all values corresponding to negative indices are zero)

$$F(z) = \mathcal{Z}\{f(kT)\} = \mathcal{Z}\{f_k\} = \sum_{k=0}^{\infty} f_k\, z^{-k}.$$

The inverse z-transform is defined as

$$f(kT) = \mathcal{Z}^{-1}\{F(z)\} = \frac{1}{2\pi j} \oint z^{k-1} F(z)\, dz,$$

where the contour encircles the region of convergence of $F(z)$ (recall that the z-transform is defined for a particular region of z values where the infinite series sum converges).

To calculate the integral is rather not easy, and relies on the following result which is based on Cauchy's residue calculus. Consider some function $G(z)$ and let z_i be the i-th pole of $G(z)$ which is of order n. The function $(z - z_i)^n G(z)$ is then analytic at z_i and as such we can obtain its Taylor series expansion as

$$(z - z_i)^n G(z) = A_{-n} + A_{-n+1}(z - z_i) + \ldots + A_{-1}(z - z_i)^{n-1} + A_0 (z - z_i)^n + \ldots$$

A_{-1} is then defined as the residue of $G(z)$ at z_i, and is denoted as $\mathrm{Res}(z_i)$. For single poles (i.e., $n = 1$) the residue is the constant term of the Taylor series expansion. According to Cauchy's residue calculus we then have that

$$\frac{1}{2\pi j} \oint G(z)\, dz = \sum_i \mathrm{Res}(z_i), \qquad \text{[Cauchy's equation]}$$

if $G(z)$ is analytic on the closed contour where the integral is defined, as well as in its interior, except from a finite number of singularities.

Proof of the inverse z-transform: We will show that the right-hand side of the inverse z-transform is indeed $f(kT)$. To achieve this consider the right-hand side of the inverse z-transform, and substitute $F(z)$ according to its definition, i.e.,

$$\begin{aligned} \frac{1}{2\pi j} \oint z^{k-1} F(z)\, dz &= \frac{1}{2\pi j} \oint z^{k-1} \sum_{\ell=0}^{\infty} f_\ell z^{-\ell}\, dz \\ &= \frac{1}{2\pi j} \sum_{\ell=0}^{\infty} f_\ell \oint z^{k-1-\ell}\, dz, \end{aligned}$$

*[Franklin et al, 1998] §4.6.2

where the integral and the summation can be exchanged as the integration contour is taken in the region where $F(z)$ is convergent.

Letting $G(z) = z^{k-1-\ell}$, three cases can be distinguished: i) if $k-1-\ell \geq 0$ then $G(z)$ has no poles inside the contour, hence there are no residue. ii) if $k-\ell < 0$ then $G(z)$ has a pole at zero of order $\ell+1-k$. This implies that $z^{\ell+1-k}G(z) = 1$ would be constant, and according to Taylor's expansion the residue A_{-1} would be zero. iii) if $\ell = k$ then $G(z)$ has a single pole at zero. This implies then that $zG(z) = 1$, but the residue A_{-1} is now 1, the constant term of the Taylor series expansion. Therefore, all terms in the summation preceding the integral would be zero but for the case $\ell = k$ where $\sum_i \mathrm{Res}(z_i) = 1$, and by Cauchy's equation with $G(z) = z^{k-1-\ell}$ we have that $\frac{1}{2\pi j}\oint z^{k-1-\ell}\,dz = 1$. Hence, the right-hand side of the inverse z-transform definition becomes

$$\frac{1}{2\pi j}\oint z^{k-1}F(z)\,dz = f(kT),$$

which is indeed the discrete-time signal $f(kT)$.

Example 12. *Let $F(z) = \frac{z}{z-1}$ (with convergence region $|z| > 1$). We will show that its inverse z-transform is*

$$f(kT) = \begin{cases} 1, & k \geq 0, \\ 0, & k < 0. \end{cases}$$

i.e., the unit step signal.
Consider the inverse z-transform

$$f_k = \mathcal{Z}^{-1}\{F(z)\} = \frac{1}{2\pi j}\oint \frac{z}{z-1}z^{k-1}\,dz = \frac{1}{2\pi j}\oint \frac{z^k}{z-1}\,dz,$$

where the contour encircles the region $|z| > 1$, where convergence for $F(z)$ is ensured.
Consider first the case $k \geq 0$. Let $G(z) = \frac{z^k}{z-1}$ and notice that it has only one single pole at one. Hence, by the Taylor series expansion of $(z-1)^n G(z)$ with $n = 1$,

$$(z-1)G(z) = z^k = 1 + k(z-1) + \ldots$$

with the residue A_{-1} being equal to 1. By Cauchy's equation we then have that $\frac{1}{2\pi j}\oint G(z)dz = 1$, which in turn implies that $f_k = 1$ for all $k \geq 0$.

Consider now the case $k < 0$. Let $G(z) = \frac{z^k}{z-1}$ and notice that it has a single pole at one and a pole at zero of order k. For the pole at one, the Taylor expansion is the same as before, and the residue is 1. For the pole at zero, by the Taylor series expansion of $z^n G(z)$ with $n = -k$ (recall k is negative),

$$z^{-k}G(z) = \frac{1}{z-1} = -1(1 + z^{-1} + z^{-2} + \ldots).$$

The residue is -1 irrespective of the value of k. Therefore, $\mathrm{Res}(1) + \mathrm{Res}(0) = 1 - 1 = 0$. As a result, by Cauchy's equation we have that $\frac{1}{2\pi j}\oint G(z)dz = 0$, which in turn implies that $f_k = 0$ for all $k < 0$.

Therefore, we have shown that the inverse z-transform of $F(z) = \frac{z}{z-1}$ is the unit step as expected from the inverse z-transform tables. However, it should be noted that knowledge of the region of convergence for the z-transform (in that case $|z| > 1$) is essential for the determination of the integral's contour, and hence the correct calculation of the inverse transform.