Probabilistic Stabilizability Certificates for a Class of Black-Box Linear Systems

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***Abstract*—We provide out-of-sample certificates on the controlled invariance property of a given set with respect to a class of black-box linear systems generated by a possibly inexact quantification of some parameters in the state-space matrices. By exploiting a set of realizations of those undetermined parameters, verifying the controlled invariance property of the given set amounts to a linear program, whose feasibility allows us to establish an a-posteriori probabilistic certificate on the controlled invariance property of such a set with respect to the unknown linear time-invariant dynamics. We apply this framework to the control of a networked system with unknown weighted graph.**

***Index Terms*—Randomized algorithms, statistical learning, linear systems.**

I. Introduction

GUARANTEEING the existence of a feedback control law capable of enforcing state constraints is essential for many control systems. A well-established technique in systems-and-control requires one to verify the controlled invariance property of a certain set, thus certifying the existence of a controller that does not allow system trajectories, initialized within the set, to escape from that set (see, e.g., [1], [2]).

In contrast with traditional model-based approaches, data-driven and learning techniques for control invariance and stabilizability problems have recently been attracting significant attention [3]–[5]. Among them, a certain line of research leverages randomized methods for (controlled) invariance set estimation and set-membership verification [6]–[14].

Specifically, a data-driven algorithm to approximate the minimal robust control invariant set w.r.t. an uncertain system, albeit without invariance guarantees for unseen dynamics, was proposed in [6]. In [7], the Koopman operator and the dynamic mode decomposition were used to reconstruct invariant sets for nonlinear systems by relying on a few data snapshots only. Following the same theme, data-driven methods to compute invariant sets with probabilistic guarantees for discrete-time (DT) black-box systems were presented in [8], [9]. By relying on partial knowledge of the system model, [10] proposed a procedure to compute probabilistic reachable sets for linear systems affected by stochastic disturbances, while the concept of stochastic invariance for control systems through probabilistic controlled invariant sets was introduced and thoroughly investigated in [11]. Randomized approaches to estimate chance-constrained sets with probabilistic guarantees, frequently encountered in control, were discussed in [12], [13]. Finally, a scenario-based set invariance verification approach for black-box systems was proposed in [14], where the observation of system trajectory snapshots allowed to compute almost-invariant sets enjoying probabilistic invariance certificates.

Similarly to [14], we investigate a scenario-based approach for the verification of the controlled invariance property of a given set. Unlike the aforementioned literature, we consider a DT linear time-invariant (LTI) system whose dynamics, described by the state-space matrices $(\bar{A}, \bar{B})$, is unknown, though belonging to a certain family $\{(A(\delta), B(\delta))\}_{\delta \in \Delta}$ due to a possibly inexact quantification of some parameters, encoded by a vector $\delta \in \Delta$ (Section III). By exploiting available realizations of δ, we propose a data-based affine policy to sample the space of feasible control inputs at the vertices of the given set whose controlled invariance property is to be verified. We are then able to translate the control invariance property verification of the given set with a prescribed affine policy into a linear program (LP) (Section IV). The feasibility of such an LP, along with known results in scenario theory [15], [16], typically characterizing decision-making problems [14], [17]–[19], allow us to establish an a-posteriori probabilistic bound on the controlled invariance property of a given set w.r.t. any LTI dynamics realized by unseen scenarios of δ, including the nominal one (Section V). We illustrate our approach on a networked, multi-agent system with edge weights in the underlying graph not deterministically known (Sections II and VI).

II. Motivating Example: Networked Multi-Agent System With Unknown Weighted Graph

To motivate the control problem addressed throughout this letter, we consider a static network of n entities that exchange information locally according to a connected and undirected graph $\mathcal{G} := (\mathcal{N}, \mathcal{E}, \mathcal{W})$ with known topology. The set $\mathcal{N} :=$

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$\{1, \ldots, n\}$ indexes the agents with scalar variable $x_i \in \mathbb{R}$, $\mathcal{E} \subseteq \{(i,j) \in \mathcal{N}^2 \mid i \neq j\}$ denotes the information flow links, while $\mathcal{W} \subseteq \mathbb{R}^{|\mathcal{E}|}$ the possible weights on the edges. We consider an instance where the agents follow a weighted agreement protocol that is also influenced by constrained external inputs $u \in \mathscr{U} \subseteq \mathbb{R}^m$ injected at m specific nodes. We can therefore split the set $\mathcal{N} = \mathcal{N}_F \cup \mathcal{N}_I$ into floating ($\mathcal{N}_F$, $n_F := |\mathcal{N}_F|$) and input nodes ($\mathcal{N}_I$, $m := |\mathcal{N}_I|$). The incidence matrix $D \in \mathbb{R}^{n \times |\mathcal{E}|}$ associated to $\mathcal{G}$ can be partitioned as $D = \mathrm{col}(D_F, D_I)$, with $D_F \in \mathbb{R}^{n_F \times |\mathcal{E}|}$ and $D_I \in \mathbb{R}^{m \times |\mathcal{E}|}$, thus leading to the following (possibly constrained) DT LTI dynamics characterizing the floating node states $x_F := \mathrm{col}((x_i)_{i \in \mathcal{N}_F}) \in \mathscr{X}_F$ [20]

$$x_F^+ = \bar{A}_F x_F + \bar{B}_F u, \tag{1}$$

where $\bar{A}_F := A_F(\bar{w}) = I_{n_F} - D_F \bar{W} D_F^\top$, $\bar{B}_F := B_F(\bar{w}) = -D_F \bar{W} D_I^\top$, $u := \mathrm{col}((x_i)_{i \in \mathcal{N}_I})$, $\bar{W} := \mathrm{diag}(\bar{w}) \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}|}$, and $\bar{w} \in \mathcal{W}$ is the vector of nominal weights associated with the links. However, especially when arising from physical modelling, such an $\bar{w}$ is not a-priori accessible, and it is typically hard to quantify exactly. Therefore, we may have available either a rough estimate of the weights on the links, or some measurements [21]–[24] in the form of *scenarios* $\{w^{(1)}, \ldots, w^{(K)}\}$, where each $w^{(j)} \in \mathcal{W}$, $j = 1, \ldots, K$, and hence a finite family of state-space matrices $\{(A_F(w^{(j)}), B_F(w^{(j)}))_{j=1}^K\} \subseteq \{(A_F(w), B_F(w))_{w \in \mathcal{W}}\}$. However, particularly when the set $\mathcal{W}$ allows for nonpositive weights, each observed scenario gives rise to an LTI weighted consensus protocols on a network as in (1), whose state evolution can be very rich, including steady-state trajectories that are synchronized, clustered, or even unstable [21], [23].

Therefore, establishing whether the unknown dynamics in (1) is stabilizable in some given set of "safe" states $\mathcal{S}_F \subseteq \mathscr{X}_F$ by means of suitable control inputs, i.e., if the closed-loop trajectories of (1) satisfy $x_F(t) \in \mathcal{S}_F$, $t \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, becomes essential. In this letter, we adopt a probabilistic treatment to answer the question: is $\mathcal{S}_F$ controlled invariant if $\bar{w}$ (and hence $(\bar{A}_F, \bar{B}_F)$) is not available a-priori, and we only have access to scenarios $\{w^{(1)}, \ldots, w^{(K)}\}$, giving rise to a set of system matrices? Specifically, we distinguish between two phases in constructing our controller:

i) The synthesis (or offline) step: only scenarios are available, but we have significant capacity for computation;
ii) The runtime (or online) step: we observe $\bar{w}$ and then apply our controller with invariance certificates for $\mathcal{S}_F$.

Note that, making available $(\bar{A}_F, \bar{B}_F)$ at runtime does not allow for standard methods requiring complicated calculations for computing invariant controllers online. Instead, we work with the best available information that we have a-priori, so that we keep all computationally intensive design efforts offline, and therefore requiring a probabilistic approach to make statements about the quality of the controller we produce.

III. Problem Formulation

We will consider general DT LTI systems in the form

$$x^+ = \bar{A}x + \bar{B}u, \tag{2}$$

where $x \in \mathscr{X} \subseteq \mathbb{R}^n$ and $u \in \mathscr{U} \subseteq \mathbb{R}^m$ are the constrained vectors of measurable state variables and control inputs, with $\mathscr{X}$ be a C-set and $\mathscr{U}$ a C-polytope [2, Ch. 3]. The system matrices $\bar{A} \in \mathbb{R}^{n \times n}$ and $\bar{B} \in \mathbb{R}^{n \times m}$ are assumed to be a-priori unknown, though belonging to a (possibly infinite) family of matrices parametrized by a vector $\delta \in \Delta \subseteq \mathbb{R}^\ell$, i.e.,

$$(\bar{A}, \bar{B}) \in \{(A(\delta), B(\delta))_{\delta \in \Delta}\}, \tag{3}$$

with $A : \mathbb{R}^\ell \to \mathbb{R}^{n \times n}$ and $B : \mathbb{R}^\ell \to \mathbb{R}^{n \times m}$. In the rest of this letter, we sometimes use $x(t)$, $t \in \mathbb{N}_0$, as opposed to x, making the time dependence explicit whenever required.

Remark 1: With the inclusion in (3) we do not assume (2) to be uncertain in the sense that it follows the DT dynamics $x(t+1) = A(\delta(t))x(t) + B(\delta(t))u(t)$, with a possibly time-varying $\delta \in \Delta$. Instead, with (3) we mean that the system in (2) evolves according to DT LTI dynamics, which however is unknown at the control synthesis step due to a possibly inexact quantification of some parameters, encoded by δ.

By following [8], [9], [14], we thus refer to (2) as a *black-box* DT LTI system since the pair $(\bar{A}, \bar{B})$, which can be associated with a realization $\bar{\delta} \in \Delta$, is not a-priori accessible.

A. Stabilizability of Discrete-Time LTI Systems

We recall some key notions in the deterministic case where the state-space matrices $(\bar{A}, \bar{B})$, as well as $\mathscr{U}$, are available.

Definition 1 (Controlled Invariance): A set $\mathcal{S} \subseteq \mathscr{X}$ is a controlled invariant w.r.t. (2) if there exists a $\mathcal{C}^1$-class feedback control law $\kappa : \mathbb{R}^n \to \mathbb{R}^m$, $\kappa(x(t)) \in \mathscr{U}$, $t \in \mathbb{N}_0$, such that, for any $x(0) \in \mathcal{S}$, the trajectory originating from (2) with $u(t) = \kappa(x(t))$ satisfies $x(t) \in \mathcal{S}$, for all $t \in \mathbb{N}$.

Next, we restate a fundamental result characterizing the controlled invariance of a C-polytope $\mathcal{S}$ w.r.t. DT LTI systems as in (2), which will be key in the rest of this letter.

Lemma 1 [2, Corollary 4.46]: A C-polytope $\mathcal{S} \subseteq \mathscr{X}$ is controlled invariant for the DT LTI system in (2) if and only if, for all $x_i \in \mathrm{vert}(\mathcal{S})$ (the set of vertices of $\mathcal{S}$), there exists a feasible control input $u \in \mathscr{U}$ such that $\bar{A}x_i + \bar{B}u \in \mathcal{S}$.

A commonly used feedback control law guaranteeing the stabilizability of DT LTI systems inside $\mathcal{S}$ is the piecewise vertex control law [25], [26]. Specifically, since any state $x \in \mathcal{S}$ can be decomposed as $x = \sum_{i \in \mathcal{V}} \gamma_i x_i$ for the N vertices $\{x_i\}_{i \in \mathcal{V}}$ of $\mathcal{S}$, $\mathcal{V} := \{1, \ldots, N\}$, with $\mathbf{1}_N^\top \gamma \leq 1$, $\gamma_i \geq 0$, $i \in \mathcal{V}$, such a control law amounts to [26, Th. 2]

$$u(t) = \sum_{i \in \mathcal{V}} \gamma_i^\star(t) u_i, \tag{4}$$

where $\gamma^\star(t) \in \mathrm{argmin}_{\gamma \in [0,1]^N} \{\mathbf{1}_N^\top \gamma \mid \sum_{i \in \mathcal{V}} \gamma_i x_i = x(t)\}$ depends on the current state $x(t)$, while $\{u_i\}_{i \in \mathcal{V}}$ are arbitrary admissible control values at the vertices of $\mathcal{S}$, $\{x_i\}_{i \in \mathcal{V}}$.

B. Stabilizability of LTI Systems With Unknown Parameters

Since the system in (2) is assumed to be a black-box, we can not directly apply the control law in (4) to stabilize it, albeit the control values at the vertices $\boldsymbol{u} := \mathrm{col}((u_i)_{i \in \mathcal{V}}) \in \mathbb{R}^{mN}$ are arbitrary in $\mathscr{U}^N$. Then, let some C-polytope $\mathcal{S} \subseteq \mathscr{X}$ be given, e.g., in the form of a set of "safe" states for (2). As discussed in Section II, we wish to provide out-of-sample certificates on the controlled invariance of $\mathcal{S}$ w.r.t. the black-box system in (2) by exploiting observed realizations of the parameter δ characterizing the matrix inclusion in (3), which may be available either from historical data (system's signatures), or generated by some probabilistic model.

Formally, we assume the parameter δ to live in some probability space $(\Delta, \mathcal{D}, \mathbb{P})$, where $\Delta \subseteq \mathbb{R}^\ell$ is the support set of δ, $\mathcal{D}$ is the associated σ-algebra and $\mathbb{P}$ is a (possibly unknown) probability measure over $\mathcal{D}$. We consider $\delta_K := \{\delta^{(j)}\}_{j\in\mathcal{K}} = \{\delta^{(1)}, \dots, \delta^{(K)}\} \in \Delta^K$, $\mathcal{K} := \{1, \dots, K\}$, as a finite collection of $K \in \mathbb{N}$ independent and identically distributed (i.i.d.) realizations of δ (also called a K-multisample). We note that any realization $\delta \in \Delta$ is associated with pair of matrices $(A(\delta), B(\delta))$, and we define the set of admissible control values at the vertices $\{x_i\}_{i\in\mathcal{V}}$ of $\mathcal{S}$ allowed by such a realization $\delta \in \Delta$ as

$$\mathcal{U}_\delta := \{\boldsymbol{u} \in \mathscr{U}^N \mid A(\delta)x_i + B(\delta)u_i \in \mathcal{S},\ \forall i \in \mathcal{V}\}. \tag{5}$$

According to Lemma 1, as long as $\mathcal{U}_\delta \neq \emptyset$, the set $\mathcal{S}$ is a controlled invariant set for the LTI system $x^+ = A(\delta)x + B(\delta)u$, which is stabilizable by means of the control law (4) with admissible control values contained in $\mathcal{U}_\delta$. Thus, aiming to establish controlled invariance certificates to previously unseen realizations of δ, we introduce the definition of *violation probability* for a generic vector of input values $\boldsymbol{u}$.

Definition 2 (Violation Probability): The violation probability associated with the input values $\boldsymbol{u} \in \mathscr{U}^N$ is given by

$$V(\boldsymbol{u}) := \mathbb{P}\{\delta \in \Delta \mid \boldsymbol{u} \notin \mathcal{U}_\delta\}. \tag{6}$$

According to Lemma 1, $V : \mathscr{U}^N \to [0, 1]$ measures the violation of the controlled invariance of the set $\mathcal{S}$ associated with input values $\boldsymbol{u}$ w.r.t. an unseen pair $(A(\delta), B(\delta))$. In other words, $V(\boldsymbol{u})$ measures the realizations $\delta \in \Delta$ such that, when these are drawn, $\boldsymbol{u}$ can not guarantee the controlled invariance of $\mathcal{S}$ w.r.t. the system induced by $(A(\delta), B(\delta))$.

IV. Dealing With the Uncertainty

Note that, given any K-multisample δ_K, solving an LP allows us to compute a vector of input values at the vertices $\{x_i\}_{i\in\mathcal{V}}$, $\boldsymbol{u}_K^\star \in \mathcal{U}_{\delta_K} := \cap_{j\in\mathcal{K}} \mathcal{U}_{\delta^{(j)}}$, such that $\boldsymbol{u}_K^\star \in \mathscr{U}^N$, and

$$\forall j \in \mathcal{K} : A(\delta^{(j)})x_i + B(\delta^{(j)})u_{i,K}^\star \in \mathcal{S},\ \forall i \in \mathcal{V}. \tag{7}$$

We therefore wish to establish an a-posteriori bound on the violation probability $V(\boldsymbol{u}_K^\star)$, to claim with high confidence that the probability $\boldsymbol{u}_K^\star$ guarantees the controlled invariance of $\mathcal{S}$ w.r.t. the family $\{(A(\delta^{(j)}), B(\delta^{(j)}))_{j\in\mathcal{K}}\} \cup \{(A(\delta), B(\delta))\}$ is above a certain value. By Lemma 1, this is equivalent to concluding that $\mathcal{S}$ is a controlled invariant set for the system in (2), with the same high confidence.

A. General Control Policies

Note the conservatism inherent in (7). For any vertex $i \in \mathcal{V}$, one would consider exactly the same admissible input value, $u_{i,K}^\star$, for all the observed K samples. To alleviate this conservativism, we introduce a policy for each vertex, namely some (possibly multi-valued) functional $\pi_i : \Delta \to \mathbb{R}^m$, which maps any realization $\delta \in \Delta$ to some input value in $\mathbb{R}^m$. In fact, according to Lemma 1, on each vertex it suffices to find an admissible control value for every observed scenario $\delta^{(j)}$, i.e., for every pair of matrices $(A(\delta^{(j)}), B(\delta^{(j)}))$, $j \in \mathcal{K}$. Given a generic sample $\delta \in \Delta$, let $\Pi_\delta := \{\boldsymbol{\pi} : \Delta \to \mathbb{R}^{mN} \mid \pi_i(\delta) \in \mathcal{U}, A(\delta)x_i + B(\delta)\pi_i(\delta) \in \mathcal{S},\ \text{for all } i \in \mathcal{V}\}$ be the set of mappings $\boldsymbol{\pi}(\cdot) := \mathrm{col}((\pi_i(\cdot))_{i\in\mathcal{V}})$ returning admissible inputs at the vertices of $\mathcal{S}$ for the pair $(A(\delta), B(\delta))$. Note in addition that $\Pi_\delta \neq \emptyset$ guarantees that $\mathcal{S}$ is controlled invariant for the DT LTI system described by the specific matrices $(A(\delta), B(\delta))$ associated with the scenario δ. However, looking for an element in Π_δ amounts to an infinite dimensional problem, as such a set contains all possible mappings $\boldsymbol{\pi}(\cdot)$.

B. Affine Control Policies

To make the problem computationally tractable, we focus on a family of mappings with finite parametrization, i.e., the affine ones. Thus, for each vertex $i \in \mathcal{V}$, we define $\pi_i(\delta) := C_i\delta + d_i$, with $C_i \in \mathbb{R}^{m\times\ell}$ and $d_i \in \mathbb{R}^m$, which leads to $\boldsymbol{\pi}(\delta) := \boldsymbol{C}\delta + \boldsymbol{d}$, where $\boldsymbol{C} := \mathrm{col}((C_i)_{i\in\mathcal{V}})$ and $\boldsymbol{d} := \mathrm{col}((d_i)_{i\in\mathcal{V}})$ belong to $\mathcal{M} := \{(\boldsymbol{C}, \boldsymbol{d}) \mid C_i \in \mathbb{R}^{m\times\ell}, d_i \in \mathbb{R}^m\}$. Then, the set of admissible affine policies for a given $\delta \in \Delta$ is $\mathcal{L}_\delta := \{(\boldsymbol{C}, \boldsymbol{d}) \in \mathcal{M} \mid C_i\delta + d_i \in \mathscr{U}, A(\delta)x_i + B(\delta)(C_i\delta + d_i) \in \mathcal{S},\ \forall i \in \mathcal{V}\} \subset \Pi_\delta$. The fact that $\mathcal{L}_\delta \neq \emptyset$ ensures that $\mathcal{S}$ is a controlled invariant for the system induced by $(A(\delta), B(\delta))$. Given K observations $\delta_K \in \Delta^K$, an optimal affine policy $(\boldsymbol{C}_K^\star, \boldsymbol{d}_K^\star) \in \mathcal{L}_{\delta_K} := \cap_{j\in\mathcal{K}} \mathcal{L}_{\delta^{(j)}}$ satisfies, for all $j \in \mathcal{K}$, $\boldsymbol{C}_K^\star\delta^{(j)} + \boldsymbol{d}_K^\star \in \mathscr{U}^N$, and

$$A(\delta^{(j)})x_i + B(\delta^{(j)})(C_{i,K}^\star\delta^{(j)} + d_{i,K}^\star) \in \mathcal{S}, \forall i \in \mathcal{V}. \tag{8}$$

For any vertex $i \in \mathcal{V}$ we now obtain a different admissible input value depending on the sample $\delta^{(j)}$ at hand, in contrast with the conservative approach in (7). Moreover, unlike the infinite dimensional problem introduced in Section IV-A, computing a pair $(\boldsymbol{C}_K^\star, \boldsymbol{d}_K^\star)$ amounts to finding a feasible solution to an LP. The C-polytopes $\mathcal{S}$ and $\mathscr{U}$ are $\mathcal{S} := \{x \in \mathbb{R}^n \mid Fx \leq \mathbf{1}_p\}$ and $\mathscr{U} := \{u \in \mathbb{R}^m \mid Hu \leq \mathbf{1}_q\}$, where $F \in \mathbb{R}^{p\times n}$ and $H \in \mathbb{R}^{q\times m}$ have full column rank [2, Sec. 3.3]. Manipulating the inclusions in (8) with $\boldsymbol{x} := \mathrm{col}((x_i)_{i\in\mathcal{V}})$ leads directly to

$$\mathcal{L}_{\delta_K} = \bigcap_{j\in\mathcal{K}} \operatorname*{argmin}_{\boldsymbol{C},\boldsymbol{d}} \{0 \mid G(\delta^{(j)})(\boldsymbol{C}\delta^{(j)} + \boldsymbol{d}) \leq l(\delta^{(j)})\}, \tag{9}$$

where $G(\delta^{(j)}) := \mathrm{col}(H \otimes I_N, FB(\delta^{(j)}) \otimes I_N)$ and $l(\delta^{(j)}) := \mathrm{col}(\mathbf{1}_{qN}, \mathbf{1}_{pN} - (FA(\delta^{(j)}) \otimes I_N)\boldsymbol{x})$. Via standard manipulations, $\mathcal{L}_{\delta_K}$ in (9) can be rewritten compactly as in (10), shown at the bottom of the page. This latter amounts to an LP, thus efficiently solvable in polynomial time, with $mN(\ell + 1)$ free variables and $(q + p)NK$ linear constraints characterized by known matrices F, H and the sample pair matrices $\{(A(\delta^{(j)}), B(\delta^{(j)}))_{j\in\mathcal{K}}\}$.

V. A-Posteriori Probabilistic Certificates of Controlled Invariance

A. Main Result

In case $\mathcal{L}_{\delta_K} \neq \emptyset$, an optimal pair $(\boldsymbol{C}_K^\star, \boldsymbol{d}_K^\star)$ may not be unique since (10) is a feasibility problem. We henceforward

$$\mathcal{L}_{\delta_K} = \operatorname*{argmin}_{\boldsymbol{C},\boldsymbol{d}} \left\{0 \;\middle|\; \mathrm{diag}((G(\delta^{(j)}))_{j\in\mathcal{K}})((\boldsymbol{C} \otimes I_K)\mathrm{col}((\delta^{(j)})_{j\in\mathcal{K}}) + \boldsymbol{d} \otimes \mathbf{1}_K) \leq \mathrm{col}((l(\delta^{(j)}))_{j\in\mathcal{K}})\right\} \tag{10}$$

assume that a tie-break rule guaranteeing the uniqueness of the solution to (10) is in place. This allows us to introduce a single-valued mapping $\Theta_K : \Delta^K \to \mathcal{M}$ that, given any $\delta_K \in \Delta^K$, satisfies $\Theta_K(\delta_K) := (\boldsymbol{C}_K^\star, \boldsymbol{d}_K^\star)$. We next recall the key definition of *support subsample* to establish our probabilistic certificate of controlled invariance for a given C-polytope $\mathcal{S}$.

Definition 3 (Support subsample, [16, Definition 2]): Given any $\delta_K \in \Delta^K$, a support subsample $S \subseteq \delta_K$ is a p-tuple of unique elements of δ_K, $S := \{\delta^{(i_1)}, \dots, \delta^{(i_p)}\}$, with $i_1 < \dots < i_p$, that gives the same solution as the original K-multisample, i.e., $\Theta_p(\delta^{(i_1)}, \dots, \delta^{(i_p)}) = \Theta_K(\delta^{(1)}, \dots, \delta^{(K)})$.

Then, let $\Upsilon_K : \Delta^K \rightrightarrows \mathcal{K}$ be any algorithm returning a p-tuple $i_1, \dots, i_p$, $i_1 < \dots < i_p$, such that $\{\delta^{(i_1)}, \dots, \delta^{(i_p)}\}$ is a support subsample for δ_K, and let $s_K := |\Upsilon_K(\delta_K)|$. In this case, a support subsample for δ_K can be identified as the subset of samples that generates a minimal representation for the polyhedral feasible set of (10). The following result characterizes the violation probability of $(\boldsymbol{C}_K^\star, \boldsymbol{d}_K^\star)$, and hence establishes a probabilistic certificate for the controlled invariance property of $\mathcal{S}$ w.r.t. the black-box system in (2).

Theorem 1: Fix $\beta \in (0, 1)$ and, for any $K \in \mathbb{N}$, let $\varepsilon : \mathcal{K} \cup \{0\} \to [0, 1]$ be a function such that $\varepsilon(K) = 1$ and $\sum_{h=0}^{K-1} \binom{K}{h} (1 - \varepsilon(h))^{K-h} = \beta$. Given any C-polytope $\mathcal{S} \subseteq \mathscr{X}$, K-multisample $\delta_K \in \Delta^K$ with associated matrices $\{(A(\delta^{(j)}), B(\delta^{(j)}))_{j\in\mathcal{K}}\}$, assume that $\mathcal{L}_{\delta_K}$ in (10) is nonempty. Then, for any Θ_K, Υ_K and $\mathbb{P}$, it holds that

$$\mathbb{P}^K\{\delta_K \in \Delta^K \mid V(\boldsymbol{C}_K^\star \delta + \boldsymbol{d}_K^\star) > \varepsilon(s_K)\} \leq \beta, \qquad (11)$$

namely, the probability that $\mathcal{S}$ is a controlled invariant set w.r.t. the black-box system in (2) is at least $1-\varepsilon(s_K)$ with confidence greater than or equal to $1 - \beta$.

Proof: Given any polyhedral C-set $\mathcal{S}$ and K-multisample $\delta_K \in \Delta^K$ with associated pairs of matrices $\{(A(\delta^{(j)}), B(\delta^{(j)}))_{j\in\mathcal{K}}\}$, assuming that $\mathcal{L}_{\delta_K} \neq \emptyset$ implies that an optimal pair $(\boldsymbol{C}_K^\star, \boldsymbol{d}_K^\star)$ solving (10) exists and, assuming some tie-break rule, it is also unique. Therefore, we have $(\boldsymbol{C}_K^\star, \boldsymbol{d}_K^\star) \in \mathcal{L}_{\delta_K}$, which clearly entails the inclusion $(\boldsymbol{C}_K^\star, \boldsymbol{d}_K^\star) \in \mathcal{L}_{\delta^{(j)}}$, for all $j \in \mathcal{K}$. By construction, this means that, for every $\delta^{(j)} \in \delta_K$, $\boldsymbol{C}_K^\star \delta^{(j)} + \boldsymbol{d}_K^\star \in \mathcal{U}_{\delta_K}$ (see (5)), and hence that $\boldsymbol{C}_K^\star \delta^{(j)} + \boldsymbol{d}_K^\star \in \mathcal{U}_{\delta^{(j)}}$, for all $j \in \mathcal{K}$. These inclusions correspond to the so-called *consistency condition* stated in [16, Assumption 1] and, together with the uniqueness of the solution, we can rely on [16, Th. 1] to obtain the probabilistic bound in (11), i.e., $\mathbb{P}^K\{\delta_K \in \Delta^K \mid V(\boldsymbol{C}_K^\star \delta + \boldsymbol{d}_K^\star) > \varepsilon(s_K)\} \leq \beta$. In view of Lemma 1, we recall that $(\boldsymbol{C}_K^\star, \boldsymbol{d}_K^\star) \in \mathcal{L}_{\delta_K}$ is a necessary and sufficient condition for the affine sampling policy to return feasible input values guaranteeing the controlled invariance property of $\mathcal{S}$ w.r.t. the observed collection of DT LTI systems originated by the pairs $\{(A(\delta^{(j)}), B(\delta^{(j)}))_{j\in\mathcal{K}}\} \subseteq \{(A(\delta), B(\delta))_{\delta\in\Delta}\}$, since $A(\delta^{(j)})x_i + B(\delta^{(j)})(\boldsymbol{C}_K^\star \delta^{(j)} + \boldsymbol{d}_K^\star) \in \mathcal{S}$, for all $i \in \mathcal{V}$, $j \in \mathcal{K}$. Thus, the bound in (11) certifies that, with confidence at least $1 - \beta$, $V(\boldsymbol{C}_K^\star \delta + \boldsymbol{d}_K^\star) = \mathbb{P}\{\delta \in \Delta \mid \boldsymbol{C}_K^\star \delta + \boldsymbol{d}_K^\star \notin \mathcal{U}_\delta\} \leq \varepsilon(s_K)$, and therefore it turns out that $\mathbb{P}\{\delta \in \Delta \mid \boldsymbol{C}_K^\star \delta + \boldsymbol{d}_K^\star \in \mathcal{U}_\delta\} \geq 1 - \varepsilon(s_K)$ with the same confidence. In view of Lemma 1, this means that the affine policy computed in (10) returns feasible input values at the vertices of $\mathcal{S}$ that guarantee the controlled invariance property of $\mathcal{S}$ w.r.t. the DT LTI system originated by the pair of matrices $(A(\delta), B(\delta))$ associated to any unseen scenario $\delta \in \Delta$. ■

Remark 2: The bound in (11) can be theoretically improved through i) a *wait-and-judge* analysis [27], where $\varepsilon(j) = 1 - t(j)$, and $t(j) \in (0, 1)$ is the unique solution to $\frac{\beta}{K+1} \sum_{h=j}^{K} \binom{h}{j} t^{h-j} - \binom{K}{j} t^{K-j} = 0$. Such a bound can be employed if a nondegeneracy assumption for (10) is imposed (see [27] for a formal definition), which is however difficult to be verified; or ii) an a-priori assessment, in case a convex tie-break rule is adopted to single-out an element from $\mathcal{L}_{\delta_K}$ [28]. This would provide a violation level $\varepsilon \in (0, 1)$ satisfying $\sum_{h=0}^{s} \binom{K}{h} \varepsilon^h (1 - \varepsilon)^{K-h} = \beta$, where s is the number of variables in (10), i.e., $mN(\ell + 1)$. However, this may lead either to an unacceptably large number of samples or, for a given number of samples, to a higher value of ε – see Section VI.

The following result characterizes in terms of probabilistic stabilizability guarantees the vertex control law in (4).

Corollary 1: Under the same conditions of Theorem 1, the probability that the vertex control law in (4), with input at vertices $\{C_{i,K}^\star \bar{\delta} + d_{i,K}^\star\}_{i\in\mathcal{V}}$, makes the given C-polytope $\mathcal{S}$ controlled invariant w.r.t. the DT LTI system in (2) is at least $1 - \varepsilon(s_K)$ with confidence greater than or equal to $1 - \beta$.

Proof: From Theorem 1, if (10) is feasible, then for any unobserved sample $\delta \in \Delta$, the probability that the policy $\boldsymbol{\pi}(\delta) = \boldsymbol{C}_K^\star \delta + \boldsymbol{d}_K^\star$ returns admissible control values at the vertices of $\mathcal{S}$ is at least $1 - \varepsilon(s_K)$ with confidence $1 - \beta$. Therefore, with the same confidence, $u(t) = \sum_{i\in\mathcal{V}} \gamma_i^\star(t)(C_{i,K}^\star \bar{\delta} + d_{i,K}^\star)$, $\gamma^\star(t) \in \operatorname{argmin}_{\gamma\in[0,1]^N} \{\mathbf{1}_N^\top \gamma \mid \sum_{i\in\mathcal{V}} \gamma_i x_i = x(t)\}$, stabilizes (2) with at least the same probability $1 - \varepsilon(s_K)$. ■

Note that Theorem 1 certifies the controlled invariance of $\mathcal{S}$ w.r.t. any DT LTI system associated to an unseen scenario of $\delta \in \Delta$. Likewise, the vertex control law in (4) enjoys the stabilizability guarantees in Corollary 1 for any unobserved sample δ, i.e., with confidence at least $1 - \beta$ and input at vertices $\{C_{i,K}^\star \delta + d_{i,K}^\star\}_{i\in\mathcal{V}}$, the control in (4) stabilizes $x^+ = A(\delta)x + B(\delta)u$ with probability at least $1 - \varepsilon(s_K)$.

B. On the Nonemptiness of $\mathcal{L}_{\delta_K}$

To generalize Theorem 1 to the case where $\mathcal{L}_{\delta_K} = \emptyset$, we can restrict Δ^K to the set of K-multisamples for which the LP in (10) is feasible, i.e., $\mathcal{F}_K := \{\delta_K \in \Delta^K \mid \mathcal{L}_{\delta_K} \neq \emptyset\}$. The bound in (11) holds then with $\mathcal{F}_K$ in place of Δ^K, implying that, with confidence at most β, if the resulting LP is feasible then the probability of violation is at least $\varepsilon(s_K)$ [15], [17]. However, without restricting the entire set Δ^K to $\mathcal{F}_K$, in case the data matrices at hand can not guarantee $\mathcal{L}_{\delta_K} \neq \emptyset$, we can not assess the controlled invariance property of $\mathcal{S}$ w.r.t. (2). In fact, the LP in (10) builds upon the specific choice of an affine sampling policy $\boldsymbol{\pi}(\delta) = \boldsymbol{C}\delta + \boldsymbol{d}$, which allows us to explore only a portion of the space of feasible inputs, $\mathscr{U}^{NK}$.

We characterize next the feasibility of the LP in (9), with $K = 1$, in terms of problem data, and then we discuss the general case $K \in \mathbb{N}$. To simplify notation, in the statement and related proof, we omit the dependency on δ in G and l. We also denote with $(P)_i$ (resp., y_i) the i-th row (element) of a matrix (vector) $P \in \mathbb{R}^{n\times m}$ ($y \in \mathbb{R}^n$). Given a set of indices $\mathcal{I} \subseteq \{1, \dots, n\}$, we indicate with $P_{\mathcal{I}}$ (resp., $y_{\mathcal{I}}$) a submatrix (subvector) obtained by selecting the rows (elements) in $\mathcal{I}$.

Lemma 2: Let $n \geq m$, $K = 1$ and $\delta \in \Delta$ be any given sample with associated pair of matrices $(A(\delta), B(\delta))$. Given any C-polytope $\mathcal{S} \subseteq \mathscr{X}$, the set $\mathcal{L}_\delta$ in (9) is nonempty if

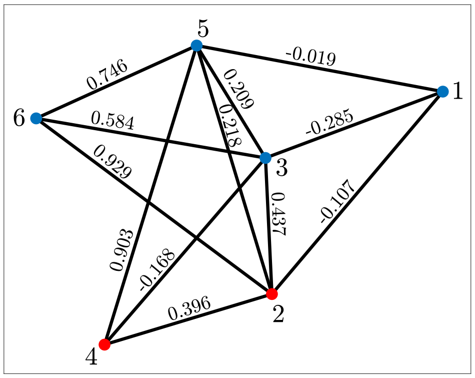

Fig. 1. Graph topology with nominal weights on the edges (black lines). The blue dots denote the floating nodes, while the red dots the input ones.

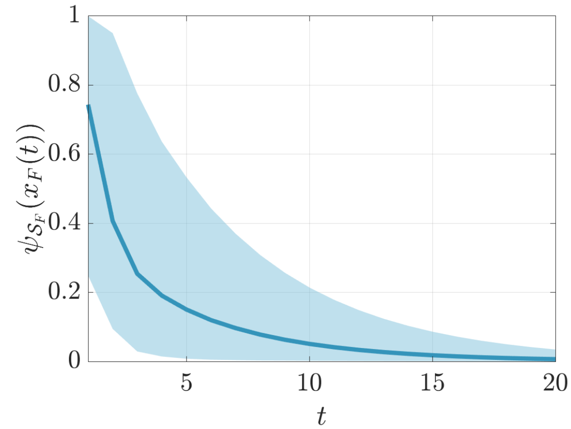

Fig. 2. Time evolution of the Minkowski function associated to the set $\mathcal{S}_F$.

and only if, for all $i \in \mathcal{V}$, there exists an invertible submatrix $G_{\mathcal{Q}\cup\mathcal{P}} \in \mathbb{R}^{m\times m}$ of G, with row indices $\mathcal{Q} \subset \{1,\dots,q\}$, $\mathcal{P} \subset \{1,\dots,p\}$, and related subvector $l_{\mathcal{Q}\cup\mathcal{P}}$ of l, such that

$$\begin{cases} (H)_k G_{\mathcal{Q}\cup\mathcal{P}}^{-1} l_{\mathcal{Q}\cup\mathcal{P}} \le 1, \ \forall k \in \bar{\mathcal{Q}} \\ (FB(\delta))_k G_{\mathcal{Q}\cup\mathcal{P}}^{-1} l_{\mathcal{Q}\cup\mathcal{P}} \le 1 - (FA(\delta)x_i)_k, \ \forall k \in \bar{\mathcal{P}} \end{cases} \tag{12}$$

with $\bar{\mathcal{Q}} := \{1,\dots,q\} \setminus \mathcal{Q}$, and $\bar{\mathcal{P}} := \{1,\dots,p\} \setminus \mathcal{P}$.

Proof: See the Appendix. ■

Extending the conditions established in Lemma 2 to the general case of $K \in \mathbb{N}$ is, however, nontrivial. In fact, from (10) it is evident that, with the same pair $(\boldsymbol{C}, \boldsymbol{d})$, one has to satisfy the inequality $G(\delta^{(j)})(\boldsymbol{C}\delta^{(j)} + \boldsymbol{d}) \le l(\delta^{(j)})$ for all $j \in \mathcal{K}$, and therefore $(\boldsymbol{C} \otimes I_K)\mathrm{col}((\delta^{(j)})_{j\in\mathcal{K}}) + \boldsymbol{d} \otimes \mathbf{1}_K \in \prod_{j\in\mathcal{K}} \mathcal{U}_{\delta^{(j)}} \subseteq \mathscr{U}^{NK}$. In terms of data matrices, the following statement provides necessary conditions only for the existence of an optimal pair $(\boldsymbol{C}_K^\star, \boldsymbol{d}_K^\star) \in \mathcal{L}_{\delta_K}$, for some $K \in \mathbb{N}$, as they essentially guarantee that $\prod_{j\in\mathcal{K}} \mathcal{U}_{\delta^{(j)}} \neq \emptyset$.

Proposition 1: Let $n \ge m$, $K \in \mathbb{N}$ and $\delta_K \in \Delta^K$ be any given K-multisample with associated pairs of matrices $\{(A(\delta^{(j)}), B(\delta^{(j)}))_{j\in\mathcal{K}}\}$. Given any C-polytope $\mathcal{S} \subseteq \mathscr{X}$, the set $\mathcal{L}_{\delta_K}$ in (10) is nonempty only if, for all $(i,j) \in \mathcal{V} \times \mathcal{K}$, there exists an invertible submatrix $G_{\mathcal{Q}\cup\mathcal{P}} \in \mathbb{R}^{m\times m}$ of $G(\delta^{(j)})$, with row indices as in Lemma 2, and subvector $l_{\mathcal{Q}\cup\mathcal{P}}$ of $l(\delta^{(j)})$, satisfying the conditions in (12).

Proof: See the Appendix. ■

Proposition 1 is only necessary for $\mathcal{L}_{\delta_K} \neq \emptyset$. In fact, if some $\tilde{\boldsymbol{u}} := \mathrm{col}((\boldsymbol{u}_j)_{j\in\mathcal{K}}) \in \mathbb{R}^{mNK}$ satisfying $\mathrm{diag}((G(\delta^{(j)}))_{j\in\mathcal{K}})\tilde{\boldsymbol{u}} \le \mathrm{col}((l(\delta^{(j)}))_{j\in\mathcal{K}})$ exists, say $\tilde{\boldsymbol{u}}^\star$, then this does not imply that we are able to find a pair $(\boldsymbol{C}_K^\star, \boldsymbol{d}_K^\star)$ such that $(\boldsymbol{C}_K^\star \otimes I_K)\mathrm{col}((\delta^{(j)})_{j\in\mathcal{K}}) + \boldsymbol{d}_K^\star \otimes \mathbf{1}_K = \tilde{\boldsymbol{u}}^\star$.

VI. Motivating Example Revisited

To illustrate our findings, we consider the graph topology represented in Fig. 1, involving $n = 6$ agents, with $\mathcal{N}_F = \{1,3,5,6\}$, $\mathcal{N}_I = \{2,4\}$, and $|\mathcal{E}| = 12$ edges with nominal weights $\bar{w}$ specified on each link. In this case, the autonomous dynamics in (1) associated with the $n_F = 4$ floating nodes (i.e., with $\bar{B}_F = 0$) is characterized by $\mathrm{eig}(\bar{A}_F) = \{-1.34, -0.01, 0.46, 0.81\}$, hence unstable. Additionally, we constraint $m = 2$ control inputs to the set $\mathscr{U} = \{u \in \mathbb{R}^2 \mid \|u\|_\infty \le 1\}$. For simplicity, $\mathcal{S}_F$ is taken as the convex hull of random points in $\pm[0.1 \ 2]$, sampled individually on each axis of $\mathbb{R}^4$, leading to a C-polytope with $N = 8$ vertices. By assuming that the entire vector of weights is not known, i.e., $\ell = |\mathcal{E}|$, we treat w as a random vector and draw $K = 600$ samples according to a uniform distribution supported on $\mathcal{W} = [0.6 \ 1.4] \times \bar{w} \subset \mathbb{R}^{12}$, i.e., a degree of uncertainty on $\bar{w}$ up to the 40%, and we compute $(\boldsymbol{C}_{600}^\star, \boldsymbol{d}_{600}^\star)$ by solving the LP in (10) on a laptop with a Quad-Core Intel Core i5 2.4 GHz CPU, 8 Gb RAM with solver Gurobi [29]. With cost function $\|\boldsymbol{C}\|_F^2 + \|\boldsymbol{d}\|^2$, 208 free variables and 96000 linear constraints, this step takes around 2.42[s]. Then, running the greedy algorithm designed in [16, Sec. II] returns a support subsample of cardinality $s_{600} = 29$, and therefore, with $\beta = 10^{-6}$, from Theorem 1 the probability that $\mathcal{S}_F$ is a controlled invariant for the floating dynamics in (1) is at least 0.7911, with confidence greater than or equal to $1 - 10^{-6}$. The function $\varepsilon(\cdot)$ in Theorem 1 is analytically obtained by splitting β evenly among the 600 terms within the summation, thus obtaining $\varepsilon(29) = 0.2089$. Note that to return the same violation level with $\beta = 10^{-6}$ the a-priori bound in Remark 2 needs $K = 1324$ samples. Conversely, with the available 600 samples, one obtains a violation level $\varepsilon = 0.259 > 0.2089$.

According to Corollary 1, the vertex control law in (4), with input at the vertices $\{C_{i,600}^\star \bar{w} + d_{i,600}^\star\}_{i\in\mathcal{V}}$, also enjoys the same certificate of $\mathcal{S}_F$. Figure 2 shows the time evolution of the Minkowski function [1, Sec. 3.3] associated to the C-polytope $\mathcal{S}_F$, i.e., $\psi_{\mathcal{S}_F}(x_F) := \min_{\lambda\ge 0}\{\lambda \mid x_F \in \lambda\mathcal{S}_F\}$. By randomly drawing 10^3 initial points in $\mathcal{S}_F$, we compute $\psi_{\mathcal{S}_F}(x_F(t))$, where $x_F(t)$ is the closed-loop trajectory originating from each initial state with control law in (4) and admissible inputs $\{C_{i,600}^\star \bar{w} + d_{i,600}^\star\}_{i\in\mathcal{V}}$. Since $\psi_{\mathcal{S}_F}(x_F(t)) \le 1$, for all $t \in \mathbb{N}$, $\mathcal{S}_F$ is not only invariant, but also a contractive set for (1).

VII. Conclusion and Outlook

By combining results in system theory and the scenario approach, we provide out-of-sample certificates on the controlled invariance property of a given set with respect to a black-box LTI system whose nominal parameters may not be determined with certainty. We propose a data-based sampling procedure to select feasible inputs at the vertices of the given set, which allows us to verify the controlled invariance property of such a set through an LP. If the LP is feasible, we establish probabilistic bounds on the controlled invariance property of the given set w.r.t. the nominal LTI system.

Directions for future work include considering different sampling policies and extending the controlled invariance property verification of given sets w.r.t. broader classes of systems, such as linear systems with polytopic uncertainty.

APPENDIX

Proof of Lemma 2: The Kronecker product in the matrix G in (9) induces a decoupled structure that allows us to focus on a single vertex $i \in \mathcal{V}$ at a time: the generalization to the entire set $\mathcal{S}$ follows readily. For some $v \in \mathrm{vert}(\mathcal{S})$, consider $G = \mathrm{col}(H, FB(\delta)) \in \mathbb{R}^{(q+p)\times m}$ and $l = \mathrm{col}(\mathbf{1}_q, \mathbf{1}_p - FA(\delta)v) \in \mathbb{R}^{q+p}$. Since H and F are full column rank matrices, we also have $\mathrm{rank}(G) = m$, as $m < q + p$, and the vertical concatenation does not alter the rank (note that $\mathrm{rank}(FB(\delta)) \leq m$, as $n \geq m$). From [30], a system of inequalities $Gu \leq l$, with $\mathrm{rank}(G) = m$, admits a solution if and only if G has a minor $\theta_m = \det(G_{\mathcal{I}}) \neq 0$ of order m, with $G_{\mathcal{I}} \in \mathbb{R}^{m\times m}$ being full rank submatrix of G with row indices $\mathcal{I} \subset \{1, \ldots, q+p\} =: \mathcal{A}$, such that

$$-\frac{1}{\theta_m}\det\left(\left[\begin{array}{c|c} G_{\mathcal{I}} & l_{\mathcal{I}} \\ \hline (G)_k & l_k \end{array}\right]\right) \leq 0,\ \forall k \in \mathcal{A} \setminus \mathcal{I}. \tag{13}$$

Since $G_{\mathcal{I}}$ is a full rank matrix, the determinant of the augmented matrix in (13) can be rewritten as $\det(G_{\mathcal{I}}) \times \det(l_k - (G)_k G_{\mathcal{I}}^{-1} l_{\mathcal{I}})$ [31]. This then implies that (13) amounts to verify $(G)_k G_{\mathcal{I}}^{-1} l_{\mathcal{I}} \leq l_k$, $\forall k \in \mathcal{A} \setminus \mathcal{I}$. Note that such inequalities guarantee the existence of some $u^\star$ that solves $Gu^\star \leq l$. In case $K = 1$, this is equivalent to guaranteeing the existence of some pair (C, d) satisfying $G(C\delta + d) \leq l$, since $C = 0$ and $d = u^\star$ is always a feasible solution. This consideration holds for each $v \in \mathrm{vert}(\mathcal{S})$, as $\boldsymbol{C} := \mathrm{col}((C_i)_{i\in\mathcal{V}})$ and $\boldsymbol{d} := \mathrm{col}((d_i)_{i\in\mathcal{V}})$. Therefore, in view of the structure of G, we can rewrite the set of row indices as $\mathcal{I} := \mathcal{Q} \cup \mathcal{P}$, with $\mathcal{Q} \subset \{1, \ldots, q\}$ and $\mathcal{P} \subset \{1, \ldots, p\}$. Then, $G_{\mathcal{I}} = \mathrm{col}(H_{\mathcal{Q}}, FB_{\mathcal{P}})$ and $l_{\mathcal{I}} = \mathrm{col}(\mathbf{1}_{|\mathcal{Q}|}, \mathbf{1}_{|\mathcal{P}|} - (FA(\delta)v)_{\mathcal{P}})$, for any $v \in \mathrm{vert}(\mathcal{S})$. Finally, the conditions in (12) follow by splitting inequalities $(G)_k G_{\mathcal{I}}^{-1} l_{\mathcal{I}} \leq l_k$, $\forall k \in \mathcal{A} \setminus \mathcal{I}$, between the two sets $\mathcal{Q}$ and $\mathcal{P}$, and noting that $(G)_k = (H)_k$ and $l_k = 1$ for any $k \in \{1, \ldots, q\} \setminus \mathcal{Q}$, while $(G)_k = (FB(\delta))_k$ and $l_k = 1 - (FA(\delta)v)_k)$, for any $k \in \{1, \ldots, p\} \setminus \mathcal{P}$.

Proof of Proposition 1: With $\tilde{\boldsymbol{u}} := \mathrm{col}((\boldsymbol{u}_j)_{j\in\mathcal{K}}) \in \mathbb{R}^{mNK}$, a solution to $\mathrm{diag}((G(\delta^{(j)}))_{j\in\mathcal{K}})\tilde{\boldsymbol{u}}^\star \leq \mathrm{col}((l(\delta^{(j)}))_{j\in\mathcal{K}})$ exists if one can find an individual $\boldsymbol{u}_j^\star \in \mathbb{R}^{mN}$ for each LP in (9). Then, the proof follows the same considerations adopted in the one for Lemma 2, for each sample $\delta^{(j)} \in \delta_K$.

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