A Scenario Approach for Non-Convex Control Design

Sergio Grammatico, Xiaojing Zhang, *Member, IEEE*, Kostas Margellos, *Member, IEEE*, Paul Goulart, *Member, IEEE*, and John Lygeros, *Fellow, IEEE*

***Abstract*—Randomized optimization is an established tool for control design with modulated robustness. While for uncertain convex programs there exist efficient randomized approaches, this is not the case for non-convex problems. Methods based on statistical learning theory are applicable to non-convex problems, but they usually are conservative in achieving the desired probabilistic guarantees. In this paper, we derive a novel scenario approach for a wide class of random non-convex programs, with a sample complexity similar to that of uncertain convex programs and with probabilistic guarantees that hold not only for the optimal solution of the scenario program, but for all feasible solutions inside a set of a-priori chosen complexity. We also address measure-theoretic issues for uncertain convex and non-convex programs. Among the family of non-convex control-design problems that can be addressed via randomization, we apply our scenario approach to stochastic model predictive control for chance constrained nonlinear control-affine systems.**

***Index Terms*—Chance constrained programs (CCPs), model predictive control (MPC), scenario program (SP).**

I. Introduction

MODERN control design often relies on the solution of an optimization problem, for instance in robust control synthesis [1], Lyapunov-based optimal control [2], [3], and Model Predictive Control (MPC) [4]. In almost all practical control applications, the data describing the plant dynamics are *uncertain*. The classic way of dealing with the uncertainty is the robust, also called min-max or worst-case, approach in which the control design has to satisfy the given specifications for *all* possible realizations of the uncertainty, for instance in robust quadratic Lyapunov synthesis problems for uncertain linear systems [5]. The worst-case approach is often formulated as a *robust* optimization problem, which is however difficult to solve in general [6]. Moreover, from an engineering perspective, robust solutions generally tend to be conservative in terms of closed-loop performance.

To reduce the conservativism of robust solutions, stochastic programming [7] offers an alternative methodology. Unlike the worst-case approach, the constraints of the problem are treated in a *probabilistic* sense via chance constraints [8], allowing for constraint violations with chosen low probability. The main issue of Chance Constrained Programs (CCPs) is that, without assumptions on the underlying probability distribution, they are in general intractable because multi-dimensional probability integrals must be computed.

Among the class of chance constrained programs, uncertain *convex* programs have received particular attention [9]; unfortunately, even for uncertain convex programs, the feasible set of a chance constrained program is in general non-convex, which makes optimization under chance constraints problematic [9, Section 1, p. 970].

An established and computationally-tractable approach to approximate chance constrained problems is the scenario approximation. A feasible solution to the CCP is found with high confidence by solving an optimization problem, called Scenario Program (SP), subject to a finite number of randomly drawn constraints (scenarios). The scenario approach is particularly effective whenever it is possible to generate samples from the uncertainty, since it does not require any further knowledge on the underlying probability distribution. From a practical point of view, this is generally the case for many control-design problems where historical data and/or predictions are available.

The scenario approach for general uncertain (so-called random) convex programs was first introduced in [10], and many control-design applications are outlined in [11]. The fundamental contribution in these works is the characterization of the number of scenarios, i.e., the sample complexity, needed to guarantee that, with high confidence, the optimal solution of the SP is a feasible solution to the original CCP. The sample complexity bound was refined in [12] where it was shown to be tight for the class of fully-supported problems, and in [13]–[15] where the concept of Helly's dimension is exploited to reduce the conservativism for non-fully-supported problems. In [13] and [16], the possibility of removing sampled constraints (sampling and discarding) is studied to improve the cost function at the price of decreased feasibility; specifically, if the constraints of the SP are removed optimally, then the solution of the SP approaches, at explicit rates, the optimal solution of the original CCP.

While feasibility and sample complexity of random convex programs are well characterized, scenario approaches for random *non-convex* programs are less developed. One family

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S. Grammatico is with the Control Systems group, Department of Electrical Engineering, Eindhoven University of Technology, The Netherlands (e-mail: grammatico@control.ee.ethz.ch; s.grammatico@tue.nl).

X. Zhang and J. Lygeros are with the Automatic Control Laboratory, ETH Zurich, Switzerland (e-mail: xiaozhan@control.ee.ethz.ch; lygeros@control.ee.ethz.ch).

K. Margellos is with the Department of Electronics, Information and Bioengineering, Polytechnic University of Milan, Milan 20122, Italy (e-mail: k.n.margellos@gmail.com).

P. Goulart is with the Department of Engineering Science, University of Oxford, Oxford, U.K. (e-mail: paul.goulart@eng.ox.ac.uk).

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of methods comes from statistical learning theory, based on the Vapnik-Chervonenkis (VC) theory [17], and it is applicable to many non-convex control-design problems [18]–[20]. Contrary to scenario approximations of uncertain convex programs [12], [13], the aforementioned methods provide probabilistic guarantees for *all feasible solutions* of the sampled program and not just for *the optimal solution*. This feature is fundamental because the global optimizer of non-convex programs is not efficiently computable in general. Moreover, having probabilistic guarantees for all feasible solutions is of interest in many applications, for instance in [21]. However, the more general probabilistic guarantees of VC theory usually require a very large number of randomly generated samples [11, Section I, p. 792], and they depend on the so-called VC dimension, in general difficult to compute or even infinite, in which case VC theory is not applicable [10, Section 1].

The aim of this paper is to propose a scenario approach for a wide class of random *non-convex* programs, with moderate sample complexity, providing probabilistic guarantees for *all feasible solutions* in a set of a-priori chosen complexity. In the spirit of [10]–[13], our results are only based on the decision complexity, while *no assumption* is made on the underlying probability structure. The main contributions of this paper with respect to the literature are listed next.

- We formulate a scenario approach for the class of random non-convex programs with (possibly) non-convex cost, deterministic (possibly) non-convex constraints, and chance constraints containing convex functions. For this class of programs, we show via a counterexample that the standard scenario approach is not directly applicable, because Helly's dimension (associated with the global optimum) can be unbounded. This motivates the development of our technique.
- We provide a sample complexity bound similar to the one of random convex programs for all feasible solutions in a set of a-priori chosen degree of complexity.
- We apply our scenario approach methodology to random non-convex programs in the presence of mixed-integer decision variables, with graceful degradation of the associated sample complexity.
- We apply our scenario approach to stochastic Model Predictive Control for nonlinear control-affine systems subject to probabilistic constraints.
- We address the measure-theoretic issues regarding the measurability of the optimal value and optimal solutions of random (convex and non-convex) programs, including the well-definiteness of the probability integrals, under minimal measurability assumptions.

The paper is structured as follows. Section II presents the technical background and the problem statement. Section III presents the main results and Section IV presents further technical extensions. Section V proposes a scenario approach for stochastic MPC of nonlinear control-affine systems. We conclude the paper in Section VI. For ease of reading, the Appendices contain: The example with unbounded Helly's dimension (Appendix A), the technical proofs (Appendix B), and the measure-theoretic results (Appendix C).

Notation: $\mathbb{R}$ and $\mathbb{Z}$ denote, respectively, the set of real and integer numbers. The notation $\mathbb{Z}[a,b]$ denotes the integer interval $\{a, a+1, \dots, b\} \subseteq \mathbb{Z}$. $\mathrm{conv}(\cdot)$ denotes the convex hull.

II. Technical Background and Problem Statement

We consider a Chance Constrained Program (CCP) with cost function $J : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, constraint function $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, constraint-violation tolerance $\epsilon \in (0,1)$, and admissible set $\mathcal{X} \subset \mathbb{R}^n$

$$\mathrm{CCP}(\epsilon) : \begin{cases} \min\limits_{x \in \mathcal{X}} \; J(x) \\ \text{s.t.}\; \mathbb{P}\left(\{\delta \in \Delta \mid g(x,\delta) \leq 0\}\right) \geq 1 - \epsilon. \end{cases} \tag{1}$$

In (1), $x \in \mathcal{X}$ is the decision variable and $\delta \in \Delta \subseteq \mathbb{R}^p$ is a random variable defined on a probability space $(\Delta, \mathcal{F}, \mathbb{P})$. The fact that $\Delta \subseteq \mathbb{R}^p$ [10, Section 1], [11, Assumption 1], [13, Section 3] simplifies the measure-theoretic arguments associated with the probability measure $\mathbb{P}$ addressed in Appendix C; however, it can be relaxed in our main results.

Throughout the paper, we make the following assumption, partially adapted from [13, Assumption 1].

Standing Assumption 1 (Regularity): The set $\mathcal{X} \subset \mathbb{R}^n$ is compact and convex. For any fixed $\bar{\delta} \in \Delta$, the mapping $x \mapsto g(x, \bar{\delta})$ is convex and lower semicontinuous. For any fixed $\bar{x} \in \mathbb{R}^n$, the mapping $\delta \mapsto g(\bar{x}, \delta)$ is measurable. The function J is lower semicontinuous. ■

The compactness assumption on $\mathcal{X}$, typical of any problem of practical interest, avoids technical difficulties by guaranteeing that any feasible problem instance attains a solution [13, Section 3.1, p. 3433]. Measurability of $g(x, \cdot)$ and lower semicontinuity of J are needed to avoid well-definiteness issues, see Appendix C for technical details.

Note that unlike the standard setting of random convex programs [10], the cost function J can be non-convex. Since our results presented later on provide probabilistic guarantees for an *entire* set, rather than for a *single* point, we next give the set-based counterpart of [11, Definitions 1, 2].

Definition 1 (Probability of Violation and Feasibility of a Set): The *probability of violation* of a set $\mathbb{X} \subseteq \mathcal{X}$ is defined as

$$V(\mathbb{X}) := \sup_{x \in \mathbb{X}} \; \mathbb{P}\left(\{\delta \in \Delta \mid g(x,\delta) > 0\}\right). \tag{2}$$

For any given $\epsilon \in (0,1)$, a set $\mathbb{X} \subseteq \mathcal{X}$ is *feasible* for $\mathrm{CCP}(\epsilon)$ in (1) if $V(\mathbb{X}) \leq \epsilon$. ■

In view of Definition 1, which accounts for the worst-case violation probability on an entire set, our developments are partially inspired by the following key statements, proved in Appendix B, regarding the violation probability of the convex hull of a given set.

Theorem 1: For given $\mathbb{X} \subseteq \mathbb{R}^n$ and $\epsilon \in [0,1]$, if $V(\mathbb{X}) \leq \epsilon$, then $V(\mathrm{conv}(\mathbb{X})) \leq (n+1)\epsilon$. For given $x_1, x_2, \dots, x_M \in \mathbb{R}^n$ and $\epsilon \in [0,1]$, if $V(\{x_1, x_2, \dots, x_M\}) \leq \epsilon$, then $V(\mathrm{conv}(\{x_1, x_2, \dots, x_M\})) \leq \min\{n+1, M\}\epsilon$. ■

An immediate consequence of Theorem 1 is that the feasibility set

$$\mathcal{X}_\epsilon := \{x \in \mathcal{X} \mid \mathbb{P}\left(\{\delta \in \Delta \mid g(x,\delta) \leq 0\}\right) \geq 1 - \epsilon\}$$

of $\mathrm{CCP}(\epsilon)$ in (1) satisfies

$$\mathcal{X}_\epsilon \subseteq \mathrm{conv}(\mathcal{X}_\epsilon) \subseteq \mathcal{X}_{(n+1)\epsilon}.$$

Associated with $\mathrm{CCP}(\epsilon)$ in (1), we consider a Scenario Program (SP) obtained from N independent and identically distributed (i.i.d.) samples $\{\bar{\delta}^{(1)}, \bar{\delta}^{(2)}, \dots, \bar{\delta}^{(N)}\}$ drawn according to $\mathbb{P}$ [10, Definition 3]. For a fixed multi-sample $\bar{\omega} := (\bar{\delta}^{(1)}, \bar{\delta}^{(2)}, \ldots, \bar{\delta}^{(N)}) \in \Delta^N$, we consider the SP

$$\mathrm{SP}[\bar{\omega}] : \begin{cases} \min\limits_{x \in \mathcal{X}} \ J(x) \\ \text{s.t. } g(x, \bar{\delta}^{(i)}) \leq 0 \ \ \forall i \in \mathbb{Z}[1, N]. \end{cases} \tag{3}$$

A. Known Results on Scenario Approximations of Chance Constraints

In [12] and [13], the case $J(x) := c^\top x$ is considered. Under the assumption that, for every multi-sample, the optimizer is unique [12, Assumption 1], [13, Assumption 2] or a suitable tie-breaking rule is adopted [10, Section 4.1] [12, Section 2.1], the optimizer mapping $x^\star(\cdot) : \Delta^N \to \mathcal{X}$ of SP$[\cdot]$ is such that

$$\mathbb{P}^N \left(\{ \omega \in \Delta^N | V(\{x^\star(\omega)\}) > \epsilon \} \right) \leq \Phi(\epsilon, n, N) := \sum\nolimits_{j=0}^{n-1} \binom{N}{j} \epsilon^j (1-\epsilon)^{N-j}. \tag{4}$$

The above bound is tight for fully-supported problems [12, Theorem 1, Equation (7)], while for non-fully-supported problems it can be improved by replacing n with the so-called Helly's dimension $\zeta \in \mathbb{Z}[1, n]$ [13, Theorem 3.3], or with an upper bound $\bar{\zeta} \in [\zeta, n]$ to Helly's dimension ζ. To satisfy the implicit bound (4) with right-hand side equal to $\beta \in (0, 1)$, it is sufficient to select a sample size [13, Corollary 5.1]

$$N \geq \frac{2}{\epsilon} \left(n - 1 + \ln\left(\frac{1}{\beta}\right) \right). \tag{5}$$

We emphasize that the inequality (4) holds only for the probability of violation of the single-valued mapping $x^\star(\cdot)$.

Although the only explicit difference between the SP in (3) and convex SPs (i.e., with $J(x) := c^\top x$) is the possibly non-convex cost J, we show in Appendix A that Helly's dimension ζ for the globally optimal value of SP in (3) can in general be unbounded. Therefore, even for the apparently simple non-convex SP in (3) it is impossible to directly apply the classic scenario approach based on Helly's theorem [22], [23].

For general non-convex programs, VC theory provides upper bounds for the quantity $\mathbb{P}^N(\{\omega \in \Delta^N | V(\mathbb{X}(\omega)) > \epsilon\})$, where $\mathbb{X}(\omega) \subseteq \mathcal{X}$ is the entire feasible set of SP$[\omega]$, see the discussions in [24, Section 3.2], [18, Sections IV, V].

In particular, [17, Theorem 8.4.1] shows that it suffices to select a sample size

$$N_{\mathrm{VC}} \geq \frac{4}{\epsilon} \left(\nu \ \ln\left(\frac{12}{\epsilon}\right) + \ln\left(\frac{2}{\beta}\right) \right) \tag{6}$$

to guarantee with confidence $1 - \beta$ that any feasible solution to SP$[\bar{\omega}]$ has probability of violation no larger than ϵ. In (6), ν is the so-called VC dimension [20, Definition 10.2], which encodes the richness of the family of functions $\{\delta \mapsto g(x, \delta) | x \in \mathcal{X}\}$ which however may be hard to estimate, or even infinite [10], [11].

III. Random Non-Convex Programs: Probabilistic Guarantees for an Entire Set

A. Main Results

We start with a preliminary intuitive statement. We consider a finite number of mappings $x_1^\star, x_2^\star, \ldots, x_M^\star : \Delta^N \to \mathcal{X}$, each one with given probabilistic guarantees, and let us upper bound their worst-case probability of violation.

Assumption 1: For a given $\epsilon \in (0, 1)$, the mappings $x_1^\star, x_2^\star, \ldots, x_M^\star : \Delta^N \to \mathcal{X}$ are such that, for all $k \in \mathbb{Z}[1, M]$, $\mathbb{P}^N(\{\omega \in \Delta^N | V(\{x_k^\star(\omega)\}) > \epsilon\}) \leq \beta_k \in (0, 1)$. ∎

Lemma 1: Consider SP$[\bar{\omega}]$ in (3). If Assumption 1 holds, then

$$\mathbb{P}^N \left(\{ \omega \in \Delta^N | V \left(\{ x_1^\star(\omega), \ldots, x_M^\star(\omega) \} \right) > \epsilon \} \right) \leq \sum_{k=1}^{M} \beta_k.$$

∎

The proof is given in Appendix B.

Note that each $x_k^\star$ can be the optimizer mapping of a convex SP and hence satisfy (4) according to [12], [13]. In such a case, with probability no smaller than $1 - M\beta$, the set $\{x_1^\star(\omega), x_2^\star(\omega), \ldots, x_M^\star(\omega)\}$ is feasible with respect to Definition 1, that is: $\mathbb{P}^N(\{\omega \in \Delta^N | V(\{x_1^\star(\omega), \ldots, x_M^\star(\omega)\}) \leq \epsilon\}) \geq 1 - M\beta$.

We may consider the meaning of Lemma 1 in view of the result in [17, Section 4.2], where the decision variable x lives in a set $\mathcal{X}$ of finite cardinality. The main difference here is that Lemma 1 instead relies on a finite number of mappings $x_k^\star(\cdot)$, rather than on a finite number of decisions. Each of these mappings is associated with a given upper bound β_k on the probability of violating the chance constraint.

We now proceed towards our main results, whose proofs are all given in Appendix B. We address the CCP(ϵ) in (1) through a family of M distinct convex SPs, each with Helly's dimension $\zeta_k \in \mathbb{Z}[1, n]$, upper bounded by some integer $\bar{\zeta} \geq \max_{k \in \mathbb{Z}[1,M]} \zeta_k$. Namely, we consider M cost vectors $c_1, c_2, \ldots, c_M \in \mathbb{R}^n$, and for each $k \in \mathbb{Z}[1, M]$, we define the following convex SP, where $\bar{\omega} := (\bar{\delta}^{(1)}, \bar{\delta}^{(2)}, \ldots, \bar{\delta}^{(N)})$

$$\mathrm{SP}_k[\bar{\omega}] : \begin{cases} \min\limits_{x \in \mathcal{C}_k \cap \mathcal{X}} \ c_k^\top x \\ \text{s.t. } g(x, \bar{\delta}^{(i)}) \leq 0 \ \ \forall i \in \mathbb{Z}[1, N]. \end{cases} \tag{7}$$

We assume that, for all $k \in \mathbb{Z}[1, M]$, SP$_k[\cdot]$ in (7) is feasible with probability 1. The additional convex constraint $x \in \mathcal{C}_k \subseteq \mathbb{R}^n$ allows us to uniformly upper bound Helly's dimension by some $\bar{\zeta} \in \mathbb{Z}[1, n]$, and its choice is hence discussed later on.

Let $x_k^\star(\bar{\omega})$ be the optimizer of SP$_k[\bar{\omega}]$ and assume that it is unique, or a suitable tie-break rule is considered [10, Section 4.1]. Note that the optimizer $x_k^\star(\bar{\omega}) \in \mathcal{X}$ is always finite, due to the compactness assumption on $\mathcal{X}$.

For all $\omega \in \Delta^N$, let us consider the convex-hull set

$$\mathbb{X}_M(\omega) := \mathrm{conv}\left(\{ x_1^\star(\omega), x_2^\star(\omega), \ldots, x_M^\star(\omega) \} \right) \tag{8}$$

where, for all $k \in \mathbb{Z}[1, M]$, $x_k^\star(\cdot)$ is the optimizer mapping of SP$_k[\cdot]$ in (7).

The role of the directions $\{c_k\}_{k=1}^M$ is to explore the decision space $\mathbb{R}^n$ to construct the set $\mathbb{X}_M$ as illustrated in Fig. 1. We in fact approximate the feasibility set of the non-convex program $\widetilde{\mathrm{SP}}[\bar{\omega}]$ in (11) with the convex hull set $\mathbb{X}_M(\bar{\omega})$ in (8). Therefore, we are interested in getting a large $\mathbb{X}_M(\bar{\omega})$ through the problems $\{\mathrm{SP}_k[\bar{\omega}]\}_{k=1}^M$. One possible approach is hence to choose linear costs $c_k^\top x$ in (7), $k = 1, \ldots, M$, as the optimal solution $x_k^\star(\bar{\omega})$ of SP$_k[\bar{\omega}]$ belongs to the boundary of the (convex) feasibility set $\mathbb{X}(\bar{\omega}) := \{x \in \mathcal{X} | g(x, \bar{\delta}^{(i)}) \leq 0 \ \forall i \in \mathbb{Z}[1, N]\}$, and so do the extreme points of the convex-hull set $\mathbb{X}_M$ in (8), as shown in Fig. 1. The selection of each direction c_k affects the actual size and shape of the convex hull set $\mathbb{X}_M(\bar{\omega})$ in (8), which however is hard to a-priori estimate. Without specific knowledge on the optimization problem, the vectors $\{c_k\}_{k=1}^M$

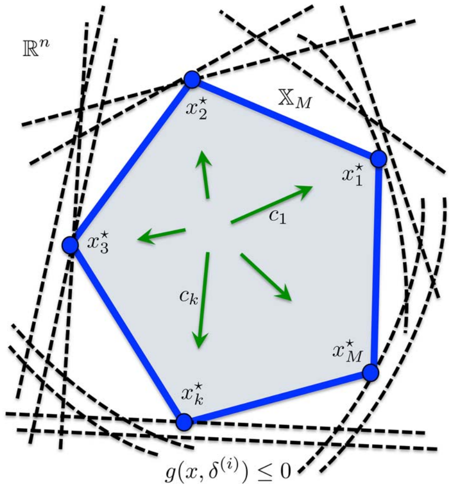

Fig. 1. The set $\mathbb{X}_M$ is the convex hull of the points $x_1^\star, x_2^\star, \ldots, x_M^\star$, where each $x_k^\star$ is the optimizer of SP_k in (7) with linear cost $c_k^\top x$.

can be chosen randomly or uniformly distributed on the unit ball.

We can now state our main result about an upper bound to the probability of violation of the constructed convex-hull set.

Theorem 2: Let $\{x_k^\star\}_{k=1}^M$ be the optimizer mappings of $\{\mathrm{SP}_k\}_{k=1}^M$ in (7), respectively, and let $\mathbb{X}_M$ be as in (8); let $\bar{\zeta} \in \mathbb{Z}[1, n]$ be a uniform upper bound to the Helly's dimensions of $\{\mathrm{SP}_k\}_{k=1}^M$. Then, for all $\epsilon \in (0, 1)$

$$\mathbb{P}^N \left(\{\omega \in \Delta^N | V(\mathbb{X}_M(\omega)) > \epsilon\} \right) \leq M \Phi \left(\frac{\epsilon}{\min\{n+1, M\}}, \bar{\zeta}, N \right). \quad (9)$$

■

Following the lines of [25, Proof of Theorem 2], we can also slightly improve the implicit bound of Theorem 2 as follows.

Corollary 1: Let $\{x_k^\star\}_{k=1}^M$ be the optimizer mappings of $\{\mathrm{SP}_k\}_{k=1}^M$ in (7), respectively, $M \geq n+1$, and let $\mathbb{X}_M$ be as in (8); let $\bar{\zeta} \in \mathbb{Z}[1, n]$ be a uniform upper bound to the Helly's dimensions of $\{\mathrm{SP}_k\}_{k=1}^M$. Then, for all $\epsilon \in (0, 1)$

$$\mathbb{P}^N \left(\{\omega \in \Delta^N | V(\mathbb{X}_M(\omega)) > \epsilon\} \right) \leq \binom{M}{n+1} \Phi \left(\epsilon, \bar{\zeta}(n+1), N \right). \quad (10)$$

■

After solving all the M SPs from (7) for the given multi-sample $\bar{\omega} \in \Delta^N$, we can solve the following approximation of $\mathrm{CCP}(\epsilon)$ in (1):

$$\tilde{\mathrm{SP}}[\bar{\omega}] : \begin{cases} \min_{x \in \mathcal{X}} \; J(x) \\ \text{s.t. } x \in \mathbb{X}_M(\bar{\omega}) \end{cases} \quad (11)$$

and explicitly establish the required sample complexity.

Corollary 2: Let $\{x_k^\star\}_{k=1}^M$ be the optimizer mappings of $\{\mathrm{SP}_k\}_{k=1}^M$ in (7), respectively, and let $\mathbb{X}_M$ be as in (8); let $\bar{\zeta} \in \mathbb{Z}[1, n]$ be a uniform upper bound to the Helly's dimensions of $\{\mathrm{SP}_k\}_{k=1}^M$. Then, for all $\epsilon, \beta \in (0, 1)$, if

$$N \geq \frac{2 \min\{n+1, M\}}{\epsilon} \left(\bar{\zeta} - 1 + \ln \left(\frac{M}{\beta} \right) \right) \quad (12)$$

then, with probability no smaller than $1 - \beta$, any feasible solution to $\tilde{\mathrm{SP}}[\bar{\omega}]$ in (11) is feasible for $\mathrm{CCP}(\epsilon)$ in (1), i.e., $\mathbb{P}^N(\{\omega \in \Delta^N | V(\mathbb{X}_M(\omega)) \leq \epsilon\}) \geq 1 - \beta$. ■

The number M of preliminary search directions determines the complexity and consequently the size of $\mathbb{X}_M$ in (8), which is the convex hull of M points. The computational cost to enlarge the approximate feasibility set $\mathbb{X}_M$ is however modest, since M affects the sample size in (12) logarithmically.

We emphasize that the probabilistic guarantees in (2) hold for *any* feasible solution to $\tilde{\mathrm{SP}}[\bar{\omega}]$ in (11), not just for the optimal solution. This is of practical importance, because $\tilde{\mathrm{SP}}[\bar{\omega}]$ is non-convex and hence computing its optimal solution is numerically intractable in general.

B. *On the Preliminary Random Convex Programs*

In this subsection we discuss the choice of the M random convex programs $\{\mathrm{SP}_k[\bar{\omega}]\}_{k=1}^M$ in (7).

The additional constraint $x \in \mathcal{C}_k$ in (7) can be used to upper bound Helly's dimension ζ_k of $\mathrm{SP}_k[\bar{\omega}]$. Many choices of $\mathcal{C}_k$ are admissible to compute (possibly conservative) estimates of the feasible region of (3) by means of $\mathbb{X}_M(\omega)$ in (8). For instance, $\mathcal{C}_k := \mathbb{R}^n$ in general only provides the upper bound $\zeta_k \leq \bar{\zeta} = n$.

The minimum upper bound on Helly's dimension for $\mathrm{SP}_k[\bar{\omega}]$ in (7), i.e., $\bar{\zeta} = 1$, is obtained whenever the decision variable x is constrained to live in a convex subspace of dimension one [14, Lemma 3.8]. For instance, assume to a-priori know a feasible point x^0 for CCP(0) in (1), which is the case in many situations of practical interest [26, Section 1.1], [27], [28]. Then, for all $k \in \mathbb{Z}[1, M]$, take $c_k \in \mathbb{R}^n$ and define

$$\mathcal{C}_k := \left\{ x^0 + \lambda c_k \in \mathbb{R}^n | \lambda \in \mathbb{R} \right\}. \quad (13)$$

With such a choice of $\mathcal{C}_k$, SP_k is equivalent to the optimization problem

$$\begin{aligned} &\min_{\lambda \in \mathbb{R}} \; -\lambda \\ &\text{s.t. } (x^0 + \lambda c_k) \in \mathcal{X}, \; g(x^0 + \lambda c_k, \delta^{(i)}) \leq 0 \;\; \forall i \in \mathbb{Z}[1, N] \end{aligned} \quad (14)$$

whose Helly's dimension is upper bounded as $\zeta_k \leq \bar{\zeta} = 1$, since the constraints are convex and the decision variable λ is 1-dimensional. In this case, the required sample size from (12) is $2 \min\{n+1, M\}/\epsilon \; \ln(M/\beta)$, which grows *linearly* in the number n of decision variables.

More generally, any selection of the convex problems $\mathrm{SP}_k[\bar{\omega}]$, and hence their associated optimizers $x_k^\star(\bar{\omega})$, for $k = 1, 2, \ldots, M$, is supported by our technical results.

IV. Extensions to More General Random Non-Convex Programs and to Sampling and Discarding

A. *Separable Non-Convex Constraints*

Let us notice that the CCP formulation in (1) implicitly includes the more general CCP

$$\mathrm{CCP}'(\epsilon) : \begin{cases} \min_{x \in \mathcal{X}} \; J(x) \\ \text{s.t. } \; \mathbb{P}(\{\delta \in \Delta | g(x, \delta) + f(x)\varphi(\delta) \leq 0\}) \geq 1 - \epsilon \\ \qquad h(x) \leq 0 \end{cases} \quad (15)$$

for possibly non-convex functions $f, h : \mathbb{R}^n \to \mathbb{R}$, $\varphi : \mathbb{R}^p \to \mathbb{R}$, at the only price of introducing one extra variable. Specifically, we can follow the lines of [29, Section 1.A, pp. 6-7] and consider $y = f(x) \in \mathcal{Y} := f(\mathcal{X})$. The probabilistic constraint then becomes $\mathbb{P}(\{\delta \in \Delta | g(x,\delta) + y\varphi(\delta) \leq 0\}) \geq 1 - \epsilon$, while the deterministic constraint becomes $\max\{h(x), |y - f(x)|\} \leq 0$. We now define the indicator function $\chi : \mathcal{X} \times \mathbb{R} \to \{0, \infty\}$ as $\chi(x,y) := 0$ if $\max\{h(x), |y - f(x)|\} \leq 0$, ∞ otherwise, so to get the CCP formulation

$$\begin{cases} \min_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \ J(x) + \chi(x,y) \\ \text{s.t. } \mathbb{P}\left(\{\delta \in \Delta | g(x,\delta) + y\varphi(\delta) \leq 0\}\right) \geq 1 - \epsilon \end{cases}$$

which has the same form as CCP(ϵ) in (1).

More generally, we can allow for "separable" non-convex probabilistic constraint of the kind

$$\mathbb{P}\left(\left\{\delta \in \Delta | g(x,\delta) + \sum_i f_i(x)\varphi_i(\delta) \leq 0\right\}\right) \geq 1 - \epsilon$$

for possibly non-convex functions $\{f_i(\cdot)\}_i$.

Furthermore, the set $\mathcal{X}$ is assumed convex without loss of generality. In fact, if $\mathcal{X}$ is not convex, let $\mathcal{X}' \supset \mathcal{X}$ be a compact convex superset of $\mathcal{X}$. Then we can define the indicator function $\chi : \mathbb{R}^n \to \{0, \infty\}$, see [29, Section 1.A, p. 6-7] as $\chi(x) := 0$ if $x \in \mathcal{X}$, ∞ otherwise. Then we define the new cost function $J + \chi$, which is lower semicontinuous as well, and finally consider the CCP with convex feasibility set $\mathcal{X}'$, namely

$$\begin{cases} \min_{x\in\mathcal{X}'} \ J(x) + \chi(x) \\ \text{s.t. } \mathbb{P}\left(\{\delta \in \Delta | g(x,\delta) \leq 0\}\right) \geq 1 - \epsilon \end{cases}$$

which has the same form as CCP(ϵ) in (1).

B. Mixed-Integer Constraints

The results in Lemma 1 and Theorem 2 can be further exploited to provide probabilistic guarantees for the following class of mixed-integer CCPs

$$\text{CCP}^{\text{m-i}}(\epsilon) : \begin{cases} \min\limits_{(x,j)\in\mathcal{X}\times\mathbb{Z}[1,L]} \ J(x) \\ \text{s.t. } \mathbb{P}\left(\{\delta \in \Delta | g_j(x,\delta) \leq 0\}\right) \geq 1 - \epsilon \end{cases} \tag{16}$$

where the functions $g_1, g_2, \ldots, g_L : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ satisfy the following assumption.

Assumption 2: For any fixed $(\bar{j}, \bar{\delta}) \in \mathbb{Z}[1,L] \times \Delta$, the mapping $x \mapsto g_{\bar{j}}(x, \bar{\delta})$ is convex and lower semicontinuous. ■

Notice that unlike [30], we also allow for possibly non-convex objective function J.

We also define the probability of violation (of any set $\mathbb{X} \subseteq \mathcal{X}$) associated with $\text{CCP}^{\text{m-i}}(\epsilon)$ in (16) as

$$V^{\text{m-i}}(\mathbb{X}) := \sup_{x\in\mathbb{X}} \ \mathbb{P}\left(\left\{\delta \in \Delta | \min_{j\in\mathbb{Z}[1,L]} g_j(x,\delta) > 0\right\}\right). \tag{17}$$

Note that, for all $j \in \mathbb{Z}[1,L]$, it holds $V^{\text{m-i}}(\mathbb{X}) \leq \sup_{x\in\mathbb{X}} \mathbb{P}(\{\delta \in \Delta | g_j(x,\delta) > 0\})$.

We can proceed similarly to Section III-A. For fixed multi-sample $\bar{\omega} \in \Delta^N$, we consider the M cost vectors $c_1, c_2, \ldots, c_M \in \mathbb{R}^n$ and the convex sets $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_M \subseteq \mathbb{R}^n$, so that, for all $(j,k) \in \mathbb{Z}[1,L] \times \mathbb{Z}[1,M]$ we define

$$\text{SP}^{\text{m-i}}_{j,k}[\bar{\omega}] : \begin{cases} \min\limits_{x\in\mathcal{C}_k\cap\mathcal{X}} \ c_k^\top x \\ \text{s.t. } g_j(x, \bar{\delta}^{(i)}) \leq 0 \ \ \forall i \in \mathbb{Z}[1,N] \end{cases} \tag{18}$$

with optimizer $x^\star_{j,k}(\bar{\omega})$. Then we can define the set $\mathbb{X}_j(\omega)$ as in (7), (8), i.e.,

$$\mathbb{X}_j(\omega) := \text{conv}\left(\{x^\star_{j,k}(\omega) | k \in \mathbb{Z}[1,M]\}\right). \tag{19}$$

If $\bar{\zeta} \in \mathbb{Z}[1,n]$ is an upper bound to Helly's dimension of the convex programs $\{\text{SP}^{\text{m-i}}_{j,k}\}_{(j,k)}$, then it follows from Theorem 2 and (17) that, for all $j \in \mathbb{Z}[1,L]$, we have

$$\begin{aligned} &\mathbb{P}^N\left(\{\omega \in \Delta^N | V^{\text{m-i}}\left(\mathbb{X}_j(\omega)\right) > \epsilon\}\right) \\ &\leq \mathbb{P}^N\left(\left\{\omega \in \Delta | \sup_{x\in\mathbb{X}_j(\omega)} \mathbb{P}\left(\{\delta \in \Delta | g_j(x,\delta)\}\right) > \epsilon\right\}\right) \\ &\leq M\Phi\left(\frac{\epsilon}{\min\{n+1, M\}}, \zeta, N\right). \end{aligned} \tag{20}$$

We can then establish the following upper bound on the probability of violation of the union of the convex-hull sets constructed above, whose proof is given in Appendix B.

Theorem 3: Suppose Assumption 2 holds. Let $\{x^\star_{j,k}\}_{j,k=1}^{L,M}$ be the optimizer mappings of $\{\text{SP}^{\text{m-i}}_{j,k}\}_{j,k=1}^{L,M}$ in (18), respectively, and let $\{\mathbb{X}_j\}_{j=1}^L$ be as in (19); let $\bar{\zeta} \in \mathbb{Z}[1,n]$ be a uniform upper bound to the Helly's dimensions of $\{\text{SP}^{\text{m-i}}_{j,k}\}_{j,k=1}^{L,M}$. Then

$$\begin{aligned} &\mathbb{P}^N\left(\{\omega \in \Delta^N | V^{\text{m-i}}\left(\cup_{j=1}^L \mathbb{X}_j(\omega)\right) > \epsilon\}\right) \\ &\qquad\leq \ LM\Phi\left(\frac{\epsilon}{\min\{n+1, M\}}, \bar{\zeta}, N\right). \end{aligned} \tag{21}$$

■

We can now approximate the $\text{CCP}^{\text{m-i}}(\epsilon)$ in (16) by

$$\tilde{\text{SP}}^{\text{m-i}}[\bar{\omega}] : \begin{cases} \min\limits_{(x,j)\in\mathcal{X}\times\mathbb{Z}[1,L]} \ J(x) \\ \text{s.t. } x \in \mathbb{X}_j(\bar{\omega}) \end{cases} \tag{22}$$

and establish the following lower bound on the sample size.

Corollary 3: Suppose Assumption 2 holds. Let $\{x^\star_{j,k}\}_{j,k=1}^{L,M}$ be the optimizer mappings of $\{\text{SP}^{\text{m-i}}_{j,k}\}_{j,k=1}^{L,M}$ in (18), respectively, and let $\{\mathbb{X}_j\}_{j=1}^L$ be as in (19); let $\bar{\zeta} \in \mathbb{Z}[1,n]$ be a uniform upper bound to the Helly's dimensions of $\{\text{SP}^{\text{m-i}}_{j,k}\}_{j,k=1}^{L,M}$. If

$$N \geq \frac{2\min\{n+1, M\}}{\epsilon}\left(\bar{\zeta} - 1 + \ln\left(\frac{LM}{\beta}\right)\right) \tag{23}$$

then, with probability no smaller than $1 - \beta$, any feasible solution to $\tilde{\text{SP}}^{\text{m-i}}[\bar{\omega}]$ in (22) is feasible for $\text{CCP}^{\text{m-i}}(\epsilon)$ in (16), i.e., $\mathbb{P}^N(\{\omega \in \Delta^N | V^{\text{m-i}}(\cup_{j=1}^L \mathbb{X}_j(\omega)) > \epsilon\}) \geq 1 - \beta$. ■

Let us comment on the sample size N given in Corollary 3, relative to the SP in (22). The formulation in (16) subsumes the one in [30]. In fact, we show that it is possible to derive a sample size N which grows linearly with the dimension d of the integer variable $y \in (\mathbb{Z}[-l/2, l/2])^d$, so that $L := (l+1)^d$ in (16). In addition to [30], [31], here we can also deal with non-convex objective functions $J(x)$ and non-convex deterministic constraints $h(x) \leq 0$ according to Section IV-A, still maintaining a sample size with logarithmic dependence on L, i.e., linear dependence on d, while [30, Theorem 3] presents an exponential dependence on d.

C. Sampling and Discarding

The problem SP_k in (7) is also suitable for a sampling-and-discarding approach [13], [16], here with the aim to reduce the optimal value of each (convex) SP_k in (7), and hence enlarge the set $\mathbb{X}_M$ in (8). Technically, the optimal objective value of the chance-constrained counterpart of SP_k in (7) can be approached at will, provided that the sampled constraints are optimally removed [16, Theorem 6.1]. While the optimal constraint removal is of combinatorial complexity, probabilistic feasibility can be obtained via any constraint removal algorithm, including greedy ones [13, Section 6.1].

The approach consists in a-priori deciding to discard r of the N samples of the uncertainty. If N and r are taken such that

$$\binom{\bar{\zeta}+r-1}{r}\Phi\left(\epsilon,\bar{\zeta}+r,N\right) = \binom{\bar{\zeta}+r-1}{r}\sum_{i=1}^{\bar{\zeta}+r-1}\binom{N}{i}\epsilon^i(1-\epsilon)^{N-i} \leq \beta \quad (24)$$

where $\bar{\zeta}\in\mathbb{Z}[1,n]$ is a uniform upper bound on the Helly's dimensions of the problems $\{\mathrm{SP}_k\}_{k=1}^M$ in (7), then, for all $k\in\mathbb{Z}[1,M]$, the optimizer mapping $x_k^\star(\cdot)$ of $\mathrm{SP}_k[\cdot]$ in (7) (where only $N-r$ constraints are imposed) is such that $\mathbb{P}^N(\{\omega\in\Delta^N| V(\{x_k^\star(\omega)\}>\epsilon)\})\leq\beta$ [13, Theorem 4.1], [16, Theorem 2.1]. Explicit bounds on the sample and removal couple (N,r) are given in [13, Section 5], [16, Section 4.3].

It then follows from (24) that, with r removals over N samples, the optimizer mappings $x_1^\star$, $x_2^\star,\ldots,x_M^\star$ satisfy Assumption 1 with $\beta_k:=\binom{\bar{\zeta}+r-1}{r}\Phi(\epsilon,\bar{\zeta}+r,N)$ for all $k\in\mathbb{Z}[1,M]$. Therefore, in view of Lemma 1, we get that the probabilistic guarantees established in Theorem 2 become

$$\mathbb{P}^N\left(\{\omega\in\Delta^N|V(\mathbb{X}_M(\omega))>\epsilon\}\right) \leq M\binom{\bar{\zeta}+r-1}{r}\Phi\left(\frac{\epsilon}{\min\{n+1,M\}},\bar{\zeta}+r,N\right).$$

Since the above inequality is with respect to $\mathbb{P}^N$, we emphasize that it is possible to remove *different* sets of r samples from each SP_k in (7). Namely, for all $k\in\mathbb{Z}[1,M]$, let $\mathcal{I}_k\subseteq\mathbb{Z}[1,M]$ be a set of indices with cardinality $|\mathcal{I}_k|=r$. Thus, we can discard the samples $\{\bar{\delta}^{(i)}|i\in\mathcal{I}_k\}$ from SP_k, possibly with $\mathcal{I}_k\neq\mathcal{I}_{k'}$ for $k\neq k'$.

V. Stochastic Model Predictive Control of Nonlinear Control-Affine Systems

In this section we extend the results of [32], [33] to uncertain *nonlinear control-affine* systems of the form

$$x^+=f(x,v)+g(x,v)\,u \quad (25)$$

where $x\in\mathbb{R}^n$ is the state variable, $u\in\mathbb{R}^m$ is the control variable, and $v\in\mathcal{V}\subseteq\mathbb{R}^p$ is the uncertain random input. We consider state constraints $x\in\mathbb{X}\subseteq\mathbb{R}^n$, where $\mathbb{X}$ is a compact convex set. We further assume the availability of i.i.d. samples $\{\bar{v}^{(1)},\bar{v}^{(2)},\ldots\}$ of the uncertain input, drawn according to a possibly unknown probability measure $\mathbb{P}$ [10, Definition 3].

For a horizon length K, let $\mathbf{u}:=[u_0;u_1;\ldots;u_{K-1}]\in\mathbb{R}^{mK}$ and $\mathbf{v}:=[v_0;v_1;\ldots;v_{K-1}]\in\mathcal{V}^K$ denote a control-input and a random-input sequence, respectively. We denote by $\phi(k;x,\mathbf{u},\mathbf{v})$ the state solution of (25) at time $k\geq 0$, starting from the initial state x, under the control-input sequence $\mathbf{u}$ and the random-input sequence $\mathbf{v}$. Likewise, given a control law $\kappa:\mathbb{R}^n\to\mathbb{R}^m$, we denote by $\phi_\kappa(k;x,\mathbf{v})$ the state solution of the system $x^+=f(x,v)+g(x,v)\kappa(x)$ at time $k\geq 0$, starting from the initial state x, under the random-input sequence $\mathbf{v}$. The solution $\phi(k;x,\mathbf{u},\mathbf{v})$, as well as $\phi_\kappa(k;x,\mathbf{v})$, is a random variable itself[1] because it depends on the random-input sequence $\mathbf{v}$.

Let $\ell:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}_{\geq 0}$ be the stage cost, and $\ell_f:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ be the terminal cost. We consider the random finite-horizon cost function

$$J(x,\mathbf{u},\mathbf{v}):=\ell_f(\phi\left(K;x,\mathbf{u},\mathbf{v}\right))+\sum_{k=0}^{K-1}\ell(\phi\left(k;x,\mathbf{u},\mathbf{v}\right),u_k). \quad (26)$$

Following [33, Section 3.1], we formulate the multi-stage Stochastic MPC (SMPC) problem:

$$\begin{cases}\min\limits_{\mathbf{u}\in\mathbb{R}^{mK}} & \mathbb{E}\left[J(x,\mathbf{u},\cdot)\right]\\ \text{s.t.} & \mathbb{P}^k\left(\{\mathbf{v}\in\mathcal{V}^k|\phi\left(k;x,\mathbf{u},\mathbf{v}\right)\in\mathbb{X}\}\right)\geq 1-\epsilon\\ & \qquad\qquad\qquad\forall k\in\mathbb{Z}[1,K].\end{cases} \quad (27)$$

For its randomized (non-convex) counterpart, let us consider three different sets of samples, indexed by the disjoint sets $\mathcal{I}_0,\mathcal{I}_1,\mathcal{I}_2\subset\mathbb{Z}[1,\infty)$, to approximate the expected cost function $\mathbb{E}[J(\cdot)]$, the first-stage constraint $f(\cdot)+g(\cdot)u\in\mathbb{X}$ and the later-stage constraints $\phi(k;\cdot)\in\mathbb{X}$, respectively

$\mathrm{SP}^{\mathrm{MPC}}[\bar{\mathbf{v}}^{(1)};\bar{\mathbf{v}}^{(2)};\ldots]$:

$$\begin{cases}\min\limits_{\mathbf{u}\in\mathbb{R}^{mK}} & \sum_{i\in\mathcal{I}_0}J(x,\mathbf{u},\bar{\mathbf{v}}^{(i)})\\ \text{s.t.} & \phi(1;x,\mathbf{u},\bar{\mathbf{v}}^{(i)})\in\mathbb{X}\quad\forall i\in\mathcal{I}_1\\ & \phi(k;x,\mathbf{u},\bar{\mathbf{v}}^{(i)})\in\mathbb{X}\quad\forall i\in\mathcal{I}_2,\ \forall k\in\mathbb{Z}[2,K].\end{cases} \quad (28)$$

The receding horizon control policy is defined as follows. For each time step, we measure the state x and let $\mathbf{u}^\star(x):=[u_0^\star;\ldots;u_{K-1}^\star](x)$ be the solution of $\mathrm{SP}^{\mathrm{MPC}}$ in (28), for some drawn samples $\{\bar{\mathbf{v}}^{(1)},\bar{\mathbf{v}}^{(2)},\ldots\}$. The control input u is set to the first element of the computed sequence, namely $u=\kappa(x):=u_0^\star(x)$, which implicitly also depends on the samples $[\bar{\mathbf{v}}^{(1)};\bar{\mathbf{v}}^{(2)};\ldots]$ extracted to construct the optimization program in (28) and hence is random itself.

We next focus on a suitable choice for the sample size, so that the average fraction of *closed-loop* constraint violations "$\{x_1\notin\mathbb{X},x_2\notin\mathbb{X},\ldots,x_t\notin\mathbb{X},\ldots\}$" is below a desired level $\epsilon\in(0,1)$. It follows from [33, Section 3] that this property can be made independent from the cardinalities of $\mathcal{I}_0$ and $\mathcal{I}_2$, i.e., on the number of samples used for the cost function and for the later stages. In fact, under proper assumptions introduced later on, the closed-loop behavior in terms of constraint violations is only influenced by the first-stage constraint, namely by the number N of samples indexed in $\mathcal{I}_1$ [33, Section 3]. Without loss of generality, let $\mathcal{I}_1:=\mathbb{Z}[1,N]$ for ease of notation. We

[1]Random solutions, both $\phi(k;x,\mathbf{u},\cdot)$ and $\phi_\kappa(k;x,\cdot)$, exist under the assumption that for all $x\in\mathbb{R}^n$, the mapping $\delta\mapsto f(x,\delta)+g(x,\delta)$ is measurable and that κ is measurable, see [34, Section 5.2] and Appendix C for technical details.

refer to [35] for a discussion on the role of $\mathcal{I}_0$ and $\mathcal{I}_2$ in terms of closed-loop performance.

In particular, we next show that our main results of Section III-A are directly applicable because the sampled nonlinear MPC program $\mathrm{SP}^{\mathrm{MPC}}$ in (28) has non-convex cost, due to the nonlinear dynamics in (25), and convex first-stage constraint. Since the program in (28) is non-convex, and hence the global optimizer is in general not computable efficiently, we use the following set-based definition of probability of violation.

Definition 2 (First-Stage Probability of Violation): For given $x \in \mathbb{X}$, $\mathbb{U}_0 \subseteq \mathbb{R}^m$, the first-stage probability of violation is

$$V^{\mathrm{MPC}}(x, \mathbb{U}_0) := \sup_{u \in \mathbb{U}_0} \mathbb{P}\left(\{v \in \mathcal{V} | f(x,v) + g(x,v)u \notin \mathbb{X}\}\right).$$

■

Analogously to Section III-A and III-B, we then consider M directions $c_1, c_2, \ldots, c_M \in \mathbb{R}^m$, and an arbitrary $\hat{u}_0 \in \mathbb{R}^m$. For instance, $\hat{u}_0$ may be a known robustly feasible solution. For all $j \in \mathbb{Z}[1, M]$, we define the following SP, where $\bar{\mathbf{v}}_0 := [\bar{v}_0^{(1)}; \bar{v}_0^{(2)}; \ldots; \bar{v}_0^{(N)}]$:

$$\mathrm{SP}_j^1[\bar{\mathbf{v}}_\mathbf{0}]: \begin{cases} \min_{\lambda \in \mathbb{R}} -\lambda \\ \text{s.t.} \quad f(x, \bar{v}_0^{(i)}) + g(x, \bar{v}_0^{(i)})\,(\hat{u}_0 + \lambda c_j) \in \mathbb{X} \\ \qquad\qquad\qquad\qquad\qquad \forall i \in \mathbb{Z}[1, N] \\ \qquad \hat{u}_0 + \lambda c_j \in \mathbb{U}. \end{cases} \tag{29}$$

Let $\lambda_j^\star(\bar{\mathbf{v}}_0)$ be the optimizer mapping of $\mathrm{SP}_j^1[\bar{\mathbf{v}}_0]$. Then, we define

$$\mathbb{U}_M(\bar{\mathbf{v}}_\mathbf{0}) := \mathrm{conv}\left(\{\hat{u}_0 + \lambda_i^\star(\bar{\mathbf{v}}_\mathbf{0}) c_i | i \in \mathbb{Z}[1, M]\}\right). \tag{30}$$

Finally, we solve the following approximation of $\mathrm{SP}^{\mathrm{MPC}}$ in (28):

$$\tilde{\mathrm{SP}}^{\mathrm{MPC}}[\bar{\mathbf{v}}^{(1)}; \bar{\mathbf{v}}^{(2)}; \ldots; \bar{\mathbf{v}}^{(N)}]:$$

$$\begin{cases} \min_{\mathbf{u} \in \mathbb{R}^{mK}} \sum_{i \in \mathcal{I}_0} J(x, \mathbf{u}, \bar{\mathbf{v}}^{(i)}) \\ \text{s.t.} \quad u_0 \in \mathbb{U}_M(\bar{\mathbf{v}}_\mathbf{0}) \\ \qquad \phi(k; x, \mathbf{u}, \bar{\mathbf{v}}^{(i)}) \in \mathbb{X} \quad \forall i \in \mathcal{I}_2,\ \forall k \in \mathbb{Z}[2, K]. \end{cases} \tag{31}$$

We can now characterize the required sample complexity for the probability of violation to be, with high confidence, below the desired level.

Theorem 4: For all $x \in \mathbb{X}$ and $j \in \mathbb{Z}[1, M]$, let $\lambda_j^\star$ be the optimizer mapping of SP_j^1 in (29), let $\mathbb{U}_M$ be as in (30), and $\epsilon, \beta \in (0, 1)$. Then

$$\mathbb{P}^N\left(\{\mathbf{v_0} \in \mathcal{V}^N | V^{\mathrm{MPC}}(x, \mathbb{U}_M(\mathbf{v_0})) > \epsilon\}\right) \le M\Phi\left(\frac{\epsilon}{\min\{m+1, M\}}, 1, N\right). \tag{32}$$

Consequently, if

$$N \ge \frac{2\min\{m+, M\}}{\epsilon} \ln\left(\frac{M}{\beta}\right) \tag{33}$$

then, with probability no smaller than $1-\beta$, any feasible solution to $\tilde{\mathrm{SP}}^{\mathrm{MPC}}$ in (31) satisfies the state constraint in (27), i.e., $\mathbb{P}^N(\{\mathbf{v_0} \in \mathcal{V}^N | V^{\mathrm{MPC}}(x, \mathbb{U}_M(\mathbf{v_0})) \le \epsilon\}) \ge 1 - \beta$. ■

The result of Theorem 4 can be exploited to characterize the *expected* closed-loop constraint violation as in [33, Theorem 14], under the recursive feasibility assumption [33, Assumption 5].

Corollary 4: Suppose that $\mathrm{SP}^{\mathrm{MPC}}$ in (28) is recursively feasible almost surely. For all $x \in \mathbb{X}$ and $\mathbf{v} := [\mathbf{v}^{(1)}; \ldots; \mathbf{v}^{(N)}] \in \mathcal{V}^{KN}$, let $\mathbf{u}(x) := [u_0(x); \ldots; u_{K-1}(x)]$ be any feasible solution to $\tilde{\mathrm{SP}}^{\mathrm{MPC}}[\mathbf{v}]$ in (31), and define $\kappa(x) := u_0(x)$. Let $\mathbb{U}_M(k; \mathbf{v})$ be defined as the set $\mathbb{U}_M$ in (30) relative to $\phi_\kappa(k; x, \mathbf{v})$. If N satisfies

$$\int_0^1 \min\left\{1, M\Phi\left(\frac{\mu}{\min\{m+1, M\}}, 1, N\right)\right\} d\mu \le \epsilon \tag{34}$$

then, for all $k \ge 0$, it holds that

$$\mathbb{E}\left[V^{\mathrm{MPC}}\left(\phi_\kappa(k; x, \cdot), \mathbb{U}_M(k; \cdot)\right)\right] := \int_{\mathcal{V}^{KN}} V^{\mathrm{MPC}}\left(\phi_\kappa(k; x, \mathbf{v}), \mathbb{U}_M(k; \mathbf{v})\right) \mathbb{P}^{KN}(\mathbf{dv}) \le \epsilon.$$

■

The meaning of Corollary 4 is that the expected closed-loop constraint violation, which can be also interpreted as time-average closed-loop constraint violation [35, Section 2.1], is upper bounded by the specified tolerance ϵ whenever the sample size N satisfies (34). A similar result was recently shown in [33, Section 4.2] for uncertain linear systems and hence here extended to the class of uncertain nonlinear control-affine systems in (25).

Numerical simulations of the proposed randomized nonlinear MPC approach are provided in [35] for a nonholonomic control-affine system, where we show the benefits with respect to randomized linear MPC.

VI. Conclusion

A. Conclusion

We have considered a scenario approach for the class of random non-convex programs with possibly non-convex cost, deterministic possibly non-convex constraints, and chance constraints containing functions with separable non-convexity. For this class of programs, Helly's dimension can be unbounded. We have derived probabilistic guarantees for all feasible solutions inside a convex set with a-priori chosen complexity, which affects the sample size logarithmically.

Our scenario approach also extends to the case with mixed-integer decision variables. We have applied our scenario approach to stochastic model predictive control for nonlinear control-affine systems with chance constraints.

Finally, we show well-definiteness of the probability integrals and measurability of the optimal value and optimal solutions of random (convex and non-convex) programs.

B. Outlook

The results in this paper can be extended in many ways. Since the probabilistic guarantees hold for any feasible solution inside a certain convex hull set, therefore it would be important to develop a scenario approach relative to the set of local minima

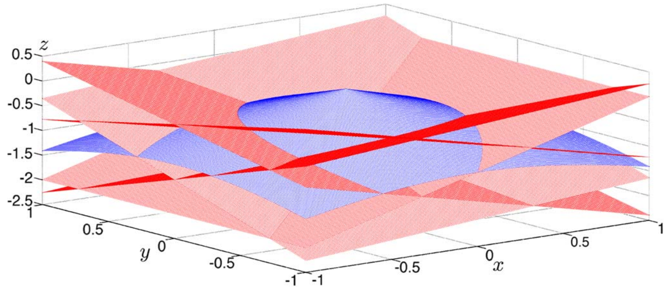

Fig. 2. The constraints of the problem $\mathrm{SP}_{\mathrm{ex}}[\bar{\omega}]$ with $N = 5$ are represented. The blue surface is the set of points such that $z = -\sqrt{x^2 + y^2}$, while the red hyperplanes are the sets of points such that $z = \cos(\bar{\delta}^{(i)})x + \sin(\bar{\delta}^{(i)})y - 1$, for $i = 1, 2, \ldots, 5$. The feasible set is the region above the plotted surfaces and the minimization direction is the vertical one, pointing down.

only. The reason is that, under mild regularity conditions, numerical solvers for non-convex programs typically ensure convergence to a local minimum. Future work should also refine the construction of the approximate feasibility set $\mathbb{X}_M$, including specific choices of the search directions $\{c_k\}_{k=1}^M$ and of the sets $\{\mathcal{C}_k\}_{k=1}^M$ to upper bound Helly's dimension of the programs $\{\mathrm{SP}_k\}_{k=1}^M$.

Our scenario approach is suitable for many non-convex control design problems, such as robust analysis and control synthesis [18]. For instance, [36] addresses typical Lyapunov control design problems via uncertain Bilinear Matrix Inequalities (BMIs), and shows comparisons with the sample complexity based on statistical learning theory [37]. Many practical control problems also rely on an uncertain non-convex optimization [19], for instance in aerospace control [38], network control [39], fault detection and isolation [40].

Appendix A
Counterexample With Unbounded Number of Support Constraints

We present an SP, derived from a CCP of the form (15), in which Helly's dimension [13, Definition 3.1] is unbounded. Namely, the number of constraints ("support constraints" [13, Definition 2.1]) needed to characterize the global optimal value equals the number N of samples

$$\mathrm{SP}_{\mathrm{ex}}[\bar{\omega}] : \begin{cases} \min_{(x,y,z)\in\mathbb{R}^3} \; z \\ \text{s.t.} \quad z \geq -\sqrt{x^2+y^2} \\ \qquad\; z \geq \cos(\bar{\delta}^{(i)})x + \sin(\bar{\delta}^{(i)})y - 1 \\ \qquad\qquad\qquad\qquad \forall i \in \mathbb{Z}[1, N]. \end{cases} \tag{35}$$

The problem can be also written in the form (3), with non-convex cost $J(x, y) := -\sqrt{x^2 + y^2}$ and non-convex constraints $-\sqrt{x^2 + y^2} \geq \cos(\bar{\delta}^{(i)})x + \sin(\bar{\delta}^{(i)})y - 1$. We use the form in (35) to visualize the optimizing direction $-z$, as shown in Fig. 2.

Let the drawn samples be $\bar{\delta}^{(i)} = (i - 1)(2\pi/N)$, for $i = 1, 2, \ldots, N$. Namely, we divide the 2π-angle into N parts, so that $\bar{\delta}^{(1)} = 0$ and $\bar{\delta}^{(i+1)} = \bar{\delta}^{(i)} + (2\pi/N)$ for all $i \in \mathbb{Z}[1, N - 1]$. We take $N \geq 5$ as $(2\pi/N) \in (0, \pi/2)$ simplifies the analysis.

We show that all the sampled constraints $z \geq \cos(\bar{\delta}^{(i)})x + \sin(\bar{\delta}^{(i)})y$, for $i = 1, 2, \ldots, N$, are support constraints, making it impossible to bound Helly's dimension by some $\zeta < N$.

We first compute the optimal value $J^\star_{\mathrm{ex}}[\bar{\omega}]$ of $\mathrm{SP}_{\mathrm{ex}}[\bar{\omega}]$ in (35). By symmetry and regularity arguments (i.e., continuity of both the objective function and the constraints in the decision variable), an optimizer $(x^\star_N, y^\star_N, z^\star_N)$ can be computed as the intersection of any two adjacent hyperplanes, say $\{(x, y, z) \in \mathbb{R}^3 | z = \cos(\bar{\delta}^{(i)})x + \sin(\bar{\delta}^{(i)})y - 1\}$ for $i = 1, 2$, and the surface $\{(x, y, z) \in \mathbb{R}^3 | z = -\sqrt{x^2 + y^2}\}$. Since $\bar{\delta}^{(1)} = 0$ and $\bar{\delta}^{(2)} = (2\pi/N) =: \theta_N \in (0, \pi/2)$, the optimal value and an optimizer can be computed by solving the system of equations

$$z = -\sqrt{x^2 + y^2} = x - 1 = \cos(\theta_N)x + \sin(\theta_N)y - 1. \tag{36}$$

From the second and the third equations in (36), we can get the solution $x^\star_N$, and then the optimal cost is $J^\star_{\mathrm{ex}}[\bar{\omega}] = z^\star_N = x^\star_N - 1$.

We then remove the sample $\bar{\delta}^{(2)} = 2\pi/N$, and hence consider the problem $\mathrm{SP}_{\mathrm{ex}}[\bar{\omega} \setminus \bar{\delta}^{(2)}]$. The optimizer is now unique and lies in the intersection of the hyperplanes $\{(x, y, z) \in \mathbb{R}^3 | z = \cos(\bar{\delta}^{(i)})x + \sin(\bar{\delta}^{(i)})y - 1\}$, for $i = 1, 3$, and the surface $\{(x, y, z) \in \mathbb{R}^3 | z = -\sqrt{x^2 + y^2}\}$. We just need to solve the system of equations (36), but with $\bar{\delta}^{(3)} := 2\theta_N = 4\pi/N$ in place of θ_N in the third equation. Therefore, we obtain almost the same solution in (36), but with $2\theta_N$ in place of θ_N. Since the optimal cost $J^\star_{\mathrm{ex}}[\bar{\omega} \setminus \bar{\delta}^{(2)}]$ is strictly smaller than $J^\star_{\mathrm{ex}}[\bar{\omega}]$ (as $x^\star_{N+1} < x^\star_N$ for all $N \geq 5$), it follows that the constraint associated with $\bar{\delta}^{(2)}$ is a support constraint. Because of the symmetry of the problem with respect to rotations around the z-axis, we conclude that all the N affine constraints $z \geq \cos(\bar{\delta}^{(i)})x + \sin(\bar{\delta}^{(i)})y - 1$, for $i = 1, 2, \ldots, N$, are support constraints as well, i.e., $J^\star_{\mathrm{ex}}[\bar{\omega} \setminus \bar{\delta}^{(i)}] < J^\star_{\mathrm{ex}}[\bar{\omega}]$ for all $i \in \mathbb{Z}[1, N]$. This proves that Helly's dimension cannot be upper bounded by any $\bar{\zeta} < N$.

Finally, in view of [12, Theorem 1], it suffices to find at least one probability measure so that the extraction of the above samples $\bar{\delta}^{(1)}, \bar{\delta}^{(2)}, \ldots, \bar{\delta}^{(N)}$ happens with non-zero probability. For instance, this holds true if $\mathbb{P}$ is such that $\mathbb{P}(\{\bar{\delta}^{(i)}\}) = 1/N$ for all $i \in \mathbb{Z}[1, N]$. Moreover, it is also possible to have a distribution about the above points $\bar{\delta}^{(1)}, \bar{\delta}^{(2)}, \ldots, \bar{\delta}^{(N)}$ that has a density, but is narrow enough to preserve the property that $J^\star[\bar{\omega} \setminus \bar{\delta}^{(2)}] < J^\star[\bar{\omega}]$.

Appendix B
Proofs

Proof of Theorem 1: By assumption, we have that $\mathbb{P}(\{\delta \in \Delta | g(x, \delta) > 0\}) \leq \epsilon$ for all $x \in \mathbb{X}$. Take any arbitrary $y \in \mathrm{conv}(\mathbb{X})$. It follows from Caratheodory's Theorem [41, Theorem 17.1] that there exist $x_1, x_2, \ldots, x_{n+1} \in \mathbb{X}$ such that $y \in \mathrm{conv}(\{x_1, x_2, \ldots, x_{n+1}\})$, i.e. $y = \sum_{i=1}^{n+1} \alpha_i x_i$ for some $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \in [0, 1]$ such that $\sum_{i=1}^{n+1} \alpha_i = 1$.

In the following inequalities, we exploit the convexity of the mapping $x \mapsto g(x, \delta)$ for each fixed $\delta \in \Delta$ from Standing

Assumption 1:

$$\begin{aligned}
&\mathbb{P}\left(\{\delta\in\Delta|g(y,\delta)>0\}\right)\\
&=\mathbb{P}\left(\left\{\delta\in\Delta|g\left(\sum_{i=1}^{n+1}\alpha_i x_i,\delta\right)>0\right\}\right)\\
&\le\mathbb{P}\left(\left\{\delta\in\Delta|\sum_{i=1}^{n+1}\alpha_i g(x_i,\delta)>0\right\}\right)\\
&\le\mathbb{P}\left(\left\{\delta\in\Delta|\max_{i\in\mathbb{Z}[1,n+1]}\alpha_i g(x_i,\delta)>0\right\}\right)\\
&=\mathbb{P}\left(\bigcup_{i=1}^{n+1}\{\delta\in\Delta|g(x_i,\delta)>0\}\right)\\
&\le\sum_{i=1}^{n+1}\mathbb{P}\left(\{\delta\in\Delta|g(x_i,\delta)>0\}\right)\le(n+1)\epsilon. \qquad (37)
\end{aligned}$$

The last inequality follows from the fact that $x_1, x_2, \ldots, x_{n+1} \in \mathcal{X}_\epsilon$. Since $y \in \mathrm{conv}(\mathbb{X})$ has been chosen arbitrarily, it follows that $V(\mathrm{conv}(\mathbb{X})) \le (n+1)\epsilon$. The proof for the case $\mathbb{X} = \{x_1, x_2, \ldots, x_M\}$ is analogous, with the only difference that if $M < n+1$, then we can just take $y = \sum_{i=1}^{M}\alpha_i x_i$, and (37) holds with $\min\{n+1, M\}$ in place of $n+1$. ∎

Proof of Lemma 1:

$$\begin{aligned}
&\mathbb{P}^N\left(\{\omega\in\Delta^N|V\left(\{x_1^\star(\omega),\ldots,x_M^\star(\omega)\}\right)>\epsilon\}\right)\\
&=\mathbb{P}^N\left(\bigcup_{j=1}^{M}\{\omega\in\Delta^N|V\left(\{x_j^\star(\omega)\}\right)>\epsilon\}\right)\\
&\le\sum_{k=1}^{M}\mathbb{P}^N\left(\{\omega\in\Delta^N|V\left(\{x_k^\star(\omega)\}\right)>\epsilon\}\right)\le\sum_{k=1}^{M}\beta_k
\end{aligned}$$

where the last inequality follows from Assumption 1. ∎

Proof of Theorem 2: It follows from Theorem 1 that, for all $\omega \in \Delta^N$, $V(\{x_1^\star(\omega), x_2^\star(\omega), \ldots, x_M^\star(\omega)\}) \le \epsilon$ implies that $V(\mathbb{X}_M(\omega)) \le \min\{n+1, M\}\epsilon$, since $\mathbb{X}_M(\omega) = \mathrm{conv}(\{x_1^\star(\omega), x_2^\star(\omega), \ldots, x_M^\star(\omega)\})$.

Therefore, we then get that $V(\mathbb{X}_M(\omega)) > \epsilon$ implies $V(\{x_1^\star(\omega), x_2^\star(\omega), \ldots, x_M^\star(\omega)\}) > (\epsilon/\min\{n+1, M\})$ for all $\omega \in \Delta^N$. This yields $\{\omega \in \Delta^N | V(\mathbb{X}_M(\omega)) > \epsilon\} \subseteq \{\omega \in \Delta^N | V(\{x_1^\star(\omega)\, x_2^\star(\omega), \ldots, x_M^\star(\omega)\}) > (\epsilon/\min\{n+1, M\})\}$ and

$$\begin{aligned}
&\mathbb{P}^N\left(\{\omega\in\Delta^N|V\left(\mathbb{X}_M(\omega)\right)>\epsilon\}\right)\le\mathbb{P}^N\Bigg(\{\omega\in\Delta^N|\\
&\qquad V\left(\{x_1^\star(\omega),\ldots,x_M^\star(\omega)\}\right)>\frac{\epsilon}{\min\{n+1,M\}}\}\Bigg). \qquad (38)
\end{aligned}$$

Since for all $k \in \mathbb{Z}[1, M]$, $x_k^\star(\cdot)$ is the optimizer mapping of $\mathrm{SP}_k[\cdot]$ in (7), and $\bar{\zeta}$ is a uniform upper bound to the Helly's dimensions of $\{\mathrm{SP}_k\}_{k=1}^M$, from [12, Theorem 1], [13, Theorem 3.3] we have that $\mathbb{P}^N(\{\omega \in \Delta^N | V(\{x_k^\star(\omega)\}) > \epsilon\}) \le \Phi(\epsilon, \bar{\zeta}, N)$. We now use Lemma 1 with $\beta_k := \Phi(\epsilon/\min\{n+1, M\}, \bar{\zeta}, N)$ for all $k \in \mathbb{Z}[1, M]$, so that

$$\begin{aligned}
&\mathbb{P}^N\big(\{\omega\in\Delta^N|V\left(\mathbb{X}_M(\omega)\right)>\epsilon\}\big)\\
&\le\mathbb{P}^N\Bigg(\Bigg\{\omega\in\Delta^N|V\left(\{x_1^\star(\omega),\ldots,x_M^\star(\omega)\}\right)>\frac{\epsilon}{\min\{n+1,M\}}\Bigg\}\Bigg)\\
&\le M\Phi\left(\frac{\epsilon}{\min\{n+1,M\}},\bar{\zeta},N\right).
\end{aligned}$$

∎

Proof of Corollary 1: It follows from Carathéodory's Theorem [41, Theorem 17.1] that, for each $\omega \in \Delta^N$, there exist the sets $\mathbb{X}_M^{(i)}(\omega) := \mathrm{conv}(\{x_k^\star(\omega) | k \in \mathcal{I}_i\})$, for $i = 1, 2, \ldots, \binom{M}{n+1}$, where each $\mathcal{I}_i$ is a set of indices of cardinality $\min\{n+1, M\}$, such that $\mathbb{X}_M(\omega) = \bigcup_{i=1}^{\binom{M}{n+1}} \mathbb{X}_M^{(i)}(\omega)$. Therefore we can write

$$\begin{aligned}
&\mathbb{P}^N\left(\{\omega\in\Delta^N|V(\mathbb{X}_M(\omega))>\epsilon\}\right)\\
&=\mathbb{P}^N\left(\left\{\omega\in\Delta^N|\sup_{x\in\mathbb{X}_M(\omega)}V(\{x\})>\epsilon\right\}\right)\\
&=\mathbb{P}^N\Bigg(\Bigg\{\omega\in\Delta^N|\max_{i\in\mathbb{Z}\left[1,\binom{M}{n+1}\right]}\\
&\qquad\times\sup_{x\in\mathbb{X}_M^{(i)}(\omega)}V(\{x\})>\epsilon\Bigg\}\Bigg)\\
&=\mathbb{P}^N\left(\bigcup_{i=1}^{\binom{M}{n+1}}\left\{\omega\in\Delta^N|V(\mathbb{X}_M^{(i)}(\omega))>\epsilon\right\}\right)\\
&\le\sum_{i=1}^{\binom{M}{n+1}}\mathbb{P}^N\left(\left\{\omega\in\Delta^N|V(\mathbb{X}_M^{(i)}(\omega))>\epsilon\right\}\right)\\
&\le\binom{M}{n+1}\mathbb{P}^N\left(\left\{\omega\in\Delta^N|V(\mathbb{X}_M^{(1)})>\epsilon\right\}\right)
\end{aligned}$$

where in the last inequality we consider the first set of indices without loss of generality, similarly to [13, Proof of Theorem 3.3, p. 3435]. It finally follows from [25, Proof of Theorem 2, Equations (22-24)] that:

$$\begin{aligned}
&\mathbb{P}^N\left(\{\omega\in\Delta^N|V(\mathbb{X}_M(\omega))>\epsilon\}\right)\\
&\le\binom{M}{n+1}\mathbb{P}^N\left(\left\{\omega\in\Delta^N|V(\mathbb{X}_M^{(1)})>\epsilon\right\}\right)\\
&\le\binom{M}{n+1}\Phi\left(\epsilon,\bar{\zeta}\min\{n+1,M\},N\right).
\end{aligned}$$

∎

Proof of Corollary 2: If follows from (10) that we need to find N such that $\Phi(\epsilon/\min\{n+1,M\},\bar{\zeta},N) \leq \beta/M$. The proof follows similarly to [13, Proof of Corollary 5.1]. ∎

Proofs of Theorem 3 and Corollary 3: The proofs are similar to the proofs of Theorem 2 and Corollary 2, respectively. ∎

Proof of Theorem 4: For each $j \in \mathbb{Z}[1,M]$, we consider the random convex problem $\mathrm{SP}_j^1[\cdot]$ in (29), with unique optimizer mapping $\lambda_j^\star(\cdot)$. Since the dimension of the decision variable is 1, i.e. $u_j^\star(\cdot) := \hat{u}_0 + \lambda_j^\star(\cdot)c_j$, it follows from [12, Theorem 1], [13, Theorem 3.3] that, for all $j \in \mathbb{Z}[1,M]$, we have

$$\mathbb{P}^N\left(\left\{\omega \in \mathcal{V}^N | V^{\mathrm{MPC}}(\{u_j^\star(\omega)\}) > \epsilon\right\}\right) \leq \Phi(\epsilon,1,N).$$

Then, from Lemma 1 we have that: $\mathbb{P}^N(\{\omega \in \mathcal{V}^N | V^{\mathrm{MPC}}\left(\{u_1^\star(\omega), u_2^\star(\omega), \ldots, u_M^\star(\omega)\}\right) > \epsilon\}) \leq M\Phi(\epsilon,1,N)$. We now notice that the CCP in (27) is of the same form of (15), with the constraints $\mathbb{P}^k(\{\mathbf{v} \in \mathcal{V}^k | \phi(k;x,\cdot,\mathbf{v}) \notin \mathbb{X}\}) \leq \epsilon$, for $k \geq 2$, in place of $h(\cdot) \leq 0$. Therefore to conclude the proof we just have to follow the steps in Section IV-A and the proof of Theorem 2 with $\{u_1^\star(\omega), \ldots u_M^\star(\omega)\}$ in place of $\{x_1^\star(\omega), \ldots x_M^\star(\omega)\}$, and finally derive the sample size N according to (34). ∎

Proof of Corollary 4: Since the sample size N satisfies (34), the proof follows from [33, Section 4.2]. ∎

Appendix C
Measurability of the Optimal Value and Optimal Solutions

In this section, we adopt the following notion of measurability from [34, Section 2]. Let $\mathcal{B}(\mathbb{R}^p)$ denote the Borel field, the subsets of $\mathbb{R}^p$ generated from open subsets of $\mathbb{R}^p$ through complements and finite countable unions. A set $F \subset \mathbb{R}^p$ is measurable if $F \in \mathcal{B}(\mathbb{R}^p)$. A set-valued mapping $M : \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ is measurable [29, Definition 14.1] if for each open set $\mathcal{O} \subset \mathbb{R}^n$ the set $M^{-1}(\mathcal{O}) := \{v \in \mathbb{R}^p | M(v) \cap \mathcal{O} \neq \varnothing\}$ is measurable. When the values of M are closed, measurability is equivalent to $M^{-1}(\mathcal{C})$ being measurable for each closed set $\mathcal{C} \in \mathbb{R}^n$ [29, Theorem 14.3]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where $\mathbb{P}$ is a probability measure on $\mathbb{R}^p$. A set $F \subset \mathbb{R}^p$ is universally measurable if it belongs to the Lebesgue completion of $\mathcal{B}(\mathbb{R}^p)$. A set-valued mapping $M : \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ is universally measurable if the set $M^{-1}(\mathcal{S})$ is universally measurable for all $\mathcal{S} \in \mathcal{B}(\mathbb{R}^n)$ [42, Section 7.1, p. 68]. If $\varphi : \mathbb{R}^p \to \mathbb{R} \cup \{\pm\infty\}$ is a (universally) measurable function, then the integral $I[\varphi] := \int_{\mathbb{R}^p} \varphi(\omega)\mathbb{P}(d\omega)$ is (nearly) well defined [29, Chapter 14, p. 643].

The following result shows (near) well definiteness of the stated probability integrals.

Theorem 5: For all $x \in \mathcal{X}$, the probability integral $\mathbb{P}(\{\delta \in \Delta | g(x,\delta) \leq 0\})$ is well defined. For any measurable set-valued mapping $\mathbb{X} : \Delta^N \rightrightarrows \mathbb{R}^n$ and $\epsilon \in (0,1)$, the probability integral $\mathbb{P}^N(\{\omega \in \Delta^N | V(\mathbb{X}(\omega)) > \epsilon\})$ is nearly well defined. ∎

Proof: From Standing Assumption 1, we have that g is a lower semicontinuous convex integrand, and hence a normal integrand [29, Proposition 14.39]. Therefore, for all $x \in \mathcal{X}$, the set $\{\delta \in \Delta | g(x,\delta) \leq 0\}$ is measurable [29, Proposition 14.33] and in turn the probabilistic measure $\mathbb{P}(\{\delta \in \Delta | g(x,\delta) \leq 0\})$ is well defined.

For the second statement, we show that, for all set-valued measurable mappings $\mathbb{X}$, the mapping $\omega \mapsto \sup_{x \in \mathbb{X}(\omega)} V(\{x\})$ is universally measurable. Since g is a normal integrand, for any finite non-negative measure μ on $\mathcal{X} \subseteq \mathbb{R}^n$, we have that g is jointly $(\mathbb{P} \otimes \mu)$-measurable [29, Corollary 14.34]. Indeed, the set $\mathcal{A} := \{(x,\delta) \in \mathcal{X} \times \Delta | g(x,\delta) \leq 0\}$ is $(\mathbb{P} \otimes \mu)$-measurable [29, Proof of Corollary 14.34], and in turn the mapping $(x,\delta) \mapsto \mathbb{1}_{\mathcal{A}}(x,\delta)$ is measurable. It then follows from Fubini's Theorem [43, Theorem 8.8 (a)] that the mapping $x \mapsto \int_\Delta \mathbb{1}_{\mathcal{A}}(x,\delta)\mathbb{P}(d\delta) = \mathbb{P}(\{\delta \in \Delta | g(x,\delta) \leq 0\})$ is measurable, and in turn $x \mapsto V(\{x\}) = 1 - \mathbb{P}(\{\delta \in \Delta | g(x,\delta) \leq 0\})$ is measurable as well [29, Proposition 14.11 (c)]. Since V is measurable, it follows from [44, Theorem 2.17 (a)] that $\omega \mapsto \sup_{x \in \mathbb{X}(\omega)} V(\{x\})$ is analytic and hence universally measurable [44, Fact 2.9]. ∎

Remark 1: According to the proof of Theorem 5, the mapping $\omega \mapsto V(\mathbb{X}(\omega))$ is *not* necessarily measurable, but just nearly measurable. However, near measurability is sufficient for the purposes of most applications, for instance in game-theory and econometrics, see [44] and the references therein.

Notice that the upper closure of V, i.e. $\bar{V}(\{x\}) := \limsup_{y \to x} V(\{y\})$, is such that the integral $\mathbb{P}^N(\{\omega \in \Delta^N | \bar{V}(\mathbb{X}(\omega)) > \epsilon\})$ is well defined. In fact, since $\bar{V}$ is upper semicontinuous by construction, $-\bar{V}$ is an autonomous, lower semicontinuous, normal integrand [29, Example 14.30]. Then it follows from [29, Example 14.32, Theorem 14.37] that the mapping $\omega \mapsto \bar{V}(\mathbb{X}(\omega)) := \sup_{x \in \mathbb{X}(\omega)} \bar{V}(\{x\})$ is measurable. ∎

We can now show the following result on the measurability of optimal value and of optimal solutions of $\mathrm{SP}[\omega]$ in (3), which means that they are random variables.

Theorem 6: Let $J^\star : \Delta^N \to \mathbb{R}$ and $\mathcal{X}^\star : \Delta^N \rightrightarrows \mathcal{X}$ be the mappings such that, for all $\omega \in \Delta^N$, $J^\star(\omega)$ and $\mathcal{X}^\star(\omega)$ are, respectively, the optimal value and the set of optimizers of $\mathrm{SP}[\omega]$ in (3). Then $J^\star$ is measurable, and $\mathcal{X}^\star$ is closed-valued and measurable. Moreover, $\mathcal{X}^\star$ admits a measurable selection, i.e., there exists a measurable mapping $x^\star : \Delta^N \to \mathcal{X}$ such that $x^\star(\omega) \in \mathcal{X}^\star(\omega)$ for all $\omega \in \Delta^N$. ∎

Proof: Since the mapping $x \mapsto g(x,\delta)$ is convex and lower semicontinuous for each δ, and the mapping $\delta \mapsto g(x,\delta)$ is measurable for each x, we have that g is a lower semicontinuous integrand and hence a normal integrand [29, Definition 14.27, Proposition 14.39]. For all $i \in \mathbb{Z}[1,N]$, we consider the lower semicontinuous convex, and hence normal [29, Proposition 14.39], integrand $g_i : \mathcal{X} \times \Delta^N \to \mathbb{R}$ defined as $g_i(x,\omega) = g_i(x,(\delta_1,\delta_2,\ldots,\delta_N)) := g(x,\delta_i)$. Then we consider the mapping $\bar{g} : \mathcal{X} \times \Delta^N \to \mathbb{R}$ defined as $\bar{g}(x,\omega) := \max_{i \in \mathbb{Z}[1,N]} g_i(x,\omega)$, which is a normal integrand because the pointwise maximum of the normal integrands g_1, $g_2, \ldots,$ g_N [29, Proposition 14.44 (a)]. We now consider the set-valued mapping $\mathcal{C} : \Delta^N \rightrightarrows \mathcal{X}$ defined as $\mathcal{C}(\omega) := \{x \in \mathcal{X} | \bar{g}(x,\omega) \leq 0\}$. Since $\bar{g}$ is a normal integrand, it follows from [29, Proposition 14.33] that the level-set mapping $\mathcal{C}$ is closed-valued and measurable. Thus, we can define the indicator integrand $\mathbb{1}_{\mathcal{C}} : \mathcal{X} \times \Delta^N \to \{0,\infty\}$ as $\mathbb{1}_{\mathcal{C}}(x,\omega) = \mathbb{1}_{\mathcal{C}(\omega)}(x) := \{0 \text{ if } x \in \mathcal{C}(\omega), \infty \text{ otherwise}\}$. Since $\mathcal{C}$ is closed-valued and measurable, the mapping $\mathbb{1}_{\mathcal{C}}$ is a normal integrand [29, Example 14.32]. Now, the problem $\mathrm{SP}[\omega]$ in (3) can be written as $\min_{x \in \mathcal{X}} c^\top x$ subject to $x \in \mathcal{C}(\omega)$, which is equivalent [29, Section 1.A] to

$\min_{x\in\mathbb{R}^n} J(x)+\mathbb{1}_{\mathcal{C}}(x,\omega)$. We notice that the mapping $(x,\omega)\mapsto \varphi(x,\omega):=J(x)+\mathbb{1}_{\mathcal{C}}(x,\omega)$ is a normal integrand as J is lower semicontinuous [29, Example 14.30, Example 14.32, Proposition 14.44 (c)]. It finally follows from [29, Theorem 14.37] that the optimal value mapping $\omega\mapsto J^\star(\omega):=\inf_{x\in\mathbb{R}^n}\varphi(x,\omega)$ is measurable; also, the set-valued mapping $\omega\mapsto\mathcal{X}^\star(\omega):=\arg\min_{x\in\mathbb{R}^n}\varphi(x,\omega)$ is closed-valued and measurable. Moreover, the set $\{\omega\in\Delta^N|\mathcal{X}^\star(\omega)\neq\varnothing\}$ is measurable, and it is possible for each $\omega\in\Delta^N$ to select a minimizing point $x^\star(\omega)$ in such a manner that the mapping $\omega\mapsto x^\star(\omega)$ is measurable [29, Corollary 14.6, Theorem 14.37]. ∎

In the following result, we show that if the set of optimizers $\mathcal{X}^\star$ of SP in (3) is not a singleton, convex and lower semicontinuous tie-break rules φ are sufficient to guarantee measurability of the optimizer $x^\star$ (whenever it is unique). Applying a tie-break rule φ basically means to solve the following program, where $J^\star(\omega)$ is the optimal value of SP$[\omega]$ in (3)

$$\mathrm{SP}_{\mathrm{t-b}}[\omega]:\begin{cases}\min_{x\in\mathcal{X}}\ \varphi(x)\\ \text{s.t.}\quad g\left(x,\delta^{(i)}\right)\le 0\ \ \forall i\in\mathbb{Z}[1,N]\\ \qquad\ J(x)\le J^\star(\omega).\end{cases}\tag{39}$$

Corollary 5: Let $\varphi:\mathbb{R}^n\to\mathbb{R}$ be a convex and lower semicontinuous function. Let $J^\star:\Delta^N\to\mathbb{R}$ and $x^\star_{\mathrm{t-b}}:\Delta^N\to\mathcal{X}$ be such that, for all $\omega\in\Delta^N$, $J^\star(\omega)$ and $x^\star_{\mathrm{t-b}}(\omega)$ are the optimal value of SP$[\omega]$ in (3) and the unique optimal solution of SP$_{\mathrm{t-b}}[\omega]$ in (39), respectively. Then $x^\star_{\mathrm{t-b}}$ is measurable. ∎

Proof: We first define the normal integrand $\bar g(x,\omega)=\bar g(x,(\delta_1,\delta_2,\ldots,\delta_N)):=\max_{i\in\mathbb{Z}[1,N]} g_i(x,\omega)=\max_{i\in\mathbb{Z}[1,N]} g(x,\delta_i)$, as in the proof of Theorem 6. Since J is lower semicontinuous, it is an autonomous integrand and hence a normal integrand [29, Example 14.30]; moreover, since $J^\star$ is measurable from Theorem 6, it is a (Carathéodory) normal integrand as well [29, Example 14.29]. Therefore also the mapping $(x,\omega)\mapsto J(x)-J^\star(\omega)$ is a normal integrand [29, Proposition 14.44 (c)], and in turn, the mapping $\bar{\bar g}(x,\omega):=\max\{\bar g(x,\omega),J(x)-J^\star(\omega)\}$ is a normal integrand as well. Then, we can just follow the proof of Theorem 6 with $\bar{\bar g}$ in place of $\bar g$. ∎

Remark 2: In (39), if J is convex and φ is strictly convex then $x^\star_{\mathrm{t-b}}(\omega)$ is the unique optimal solution of SP$_{\mathrm{t-b}}[\omega]$. ∎

We finally show that the convex hull of measurable singletons is measurable, so that $\mathbb{P}^N(\{\omega\in\Delta^N|V(\mathbb{X}_M(\omega))>\epsilon\})$ is well defined from Theorem 5.

Corollary 6: The set-valued mapping $\mathbb{X}_M$ in (8) is measurable. ∎

Proof: According to Theorem 6 and Remark 2, the unique optimal solutions $x_1^\star,\ldots,x_M^\star$, respectively of SP$_1,\ldots,SP_M$, are measurable mappings. Then the proof directly follows as $\mathbb{X}_M$ is the convex-hull set-valued mapping of a countable union of measurable mappings [29, Proposition 114.11 (b), Example 14.12 (a)]. ∎

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References

[1] P. Apkarian and H. D. Tuan, "Parameterized LMIs in control theory," *SIAM J. Control Optim.*, vol. 38, no. 4, pp. 1241–1264, 2000.

[2] D. P. Bertsekas, *Dynamic programming and optimal control*. Belmont, MA, USA: Athena Scientific, 2005.

[3] R. W. Beard, G. N. Saridis, and J. T. Wen, "Galerkin approximations of the generalized Hamilton-Jacobi-Bellman equation," *Automatica*, vol. 33, no. 12, pp. 2159–2177, 1997.

[4] D. Q. Mayne, "Model predictive control: Recent developments and future promise," *Automatica*, vol. 50, pp. 2967–2986, 2014.

[5] A. Ben-Tal and A. Nemirovski, "On tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty," *SIAM J. Optim.*, vol. 12, no. 3, pp. 811–833, 2002.

[6] A. Ben-Tal and A. Nemirovski, "Robust convex optimization," *Math. Oper. Res.*, vol. 23, no. 4, pp. 769–805, 1998.

[7] A. Prékopa, *Stochastic Programming*. New York: Springer, 1995, ser. Mathematics and Its Applications.

[8] L. B. Miller and H. Wagner, "Chance-constrained programming with joint constraints," *Oper. Res.*, vol. 13, pp. 930–945, 1965.

[9] A. Nemirovski and A. Shapiro, "Convex approximations of chance constrained programs," *SIAM J. Optim.*, vol. 17, no. 4, pp. 969–996, 2006.

[10] G. Calafiore and M. C. Campi, "Uncertain convex programs: Randomized solutions and confidence levels," *Math. Programming*, vol. 102, no. 1, pp. 25–46, 2005.

[11] G. Calafiore and M. C. Campi, "The scenario approach to robust control design," *IEEE Trans. Autom. Control*, vol. 51, no. 5, pp. 742–753, 2006.

[12] M. C. Campi and S. Garatti, "The exact feasibility of randomized solutions of robust convex programs," *SIAM J. Optim.*, vol. 19, no. 3, pp. 1211–1230, 2008.

[13] G. C. Calafiore, "Random convex programs," *SIAM J. Optim.*, vol. 20, no. 6, pp. 3427–3464, 2010.

[14] G. Schildbach, L. Fagiano, and M. Morari, "Randomized solutions to convex programs with multiple chance constraints," *SIAM J. Optim.*, vol. 23, no. 4, pp. 2479–2501, 2013.

[15] X. Zhang, S. Grammatico, G. Schildbach, P. Goulart, and J. Lygeros, "On the sample size of random convex programs with structured dependence on the uncertainty," *Automatica*, vol. 60, no. 10, pp. 182–188, 2015.

[16] M. C. Campi and S. Garatti, "A sampling-and-discarding approach to chance-constrained optimization: Feasibility and optimality," *J. Optim. Theory Appl.*, vol. 148, pp. 257–280, 2011.

[17] M. Anthony and N. Biggs, *Computational Learning Theory*. Cambridge, U.K.: Cambridge Tracts in Theoretical Computer Science, 1992.

[18] T. Alamo, R. Tempo, and E. F. Camacho, "Randomized strategies for probabilistic solutions of uncertain feasibility and optimization problems," *IEEE Trans. Autom. Control*, vol. 54, no. 11, pp. 2545–2559, Nov. 2009.

[19] G. Calafiore, F. Dabbene, and R. Tempo, "Research on probabilistic methods for control system design," *Automatica*, vol. 47, pp. 1279–1293, 2011.

[20] R. Tempo, G. Calafiore, and F. Dabbene, *Randomized Algorithms for Analysis and Control of Uncertain Systems, 2nd ed.*. New York: Springer, 2013.

[21] G. Calafiore, M. C. Campi, and L. E. Ghaoui, "Identification of reliable predictor models for unknown systems: A data-consistency approach based on learning theory," in *IFAC World Congress*, Barcelona, Spain, 2002.

[22] S. Floyd and M. Warmuth, "Sample compression, learnability, the Vapnik-Charvonenkis dimension," *Machine learning*, pp. 1–36, 1995.

[23] V. L. Levin, "Application of E. Helly's theorem to convex programming, problems of best approximation and related questions," *Math. USSR Sbornik*, vol. 8, no. 2, pp. 235–247, 1969.

[24] E. Erdogan and G. Iyengar, "Ambiguous chance constrained problems and robust optimization," *Mathematical Programming*, vol. 107, pp. 37–61, 2006.

[25] G. C. Calafiore and D. Lyons, "Random convex programs for distributed multi-agent consensus," in *Proc. IEEE Eur. Control Conf.*, Zurich, Switzerland, 2013, pp. 250–255.

[26] A. Caré, S. Garatti, and M. Campi, "FAST: An algorithm for the scenario approach with reduced sample complexity," in *IFAC World Congress*, Milano, Italy, 2011, pp. 9236–9241.

[27] M. Campi, G. Calafiore, and S. Garatti, "Interval predictor models: Identification and reliability," *Automatica*, vol. 45, pp. 382–392, 2009.

[28] M. Campi, S. Garatti, and M. Prandini, "The scenario approach for systems and control design," *Annu. Rev. Control*, vol. 33, no. 2, pp. 149–157, 2009.

[29] R. T. Rockafellar and R. J. B. Wets, *Variational Analysis*. New York: Springer, 1998.

[30] G. C. Calafiore, D. Lyons, and L. Fagiano, "On mixed-integer random convex programs," in *Proc. IEEE Conf. Decision Control*, Maui, HI, USA, 2012, pp. 3508–3513.

[31] P. Mohajerin Esfahani, T. Sutter, and J. Lygeros, "Performance bounds for the scenario approach and an extension to a class of non-convex programs," *IEEE Trans. Autom. Control*, vol. 60, pp. 46–58, 2015.

[32] G. C. Calafiore and L. Fagiano, "Robust MPC via scenario optimization," *IEEE Trans. Autom. Control*, vol. 58, pp. 219–224, 2012.

[33] G. Schildbach, L. Fagiano, C. Frei, and M. Morari, "The scenario approach for stochastic model predictive control with bounds on closed-loop constraint violations," *Automatica*, vol. 50, pp. 3009–3018, 2014.

[34] S. Grammatico, A. Subbaraman, and A. Teel, "Discrete-time stochatic control systems: A continuous Lyapunov function implies robustness to strictly causal perturbations," *Automatica*, vol. 49, pp. 2939–2952, 2013.

[35] X. Zhang, S. Grammatico, K. Margellos, P. Goulart, and J. Lygeros, "Randomized nonlinear MPC for uncertain control-affine systems with bounded closed-loop constraint violations," in *IFAC World Congress*, Cape Town, South Africa, 2014, pp. 1649–1654.

[36] S. Grammatico, X. Zhang, K. Margellos, P. Goulart, and J. Lygeros, "A scenario approach to non-convex control design: Preliminary probabilistic guarantees," in *Proc. IEEE Amer. Control Conf.*, Portland, Oregon, USA, 2014, pp. 3431–3436.

[37] M. Chamanbaz, F. Dabbene, R. Tempo, V. Venkataramanan, and Q.-G. Wang, "A statistical learning theory approach for uncertain linear and bilinear matrix inequalities," *Automatica*, vol. 50, pp. 1617–1625, 2014.

[38] Q. Wang and R. F. Stengel, "Robust nonlinear flight control of a high performance aircraft," *IEEE Trans. Control Syst. Technol.*, vol. 13, pp. 15–26, 2005.

[39] T. Alpcan, T. Basar, and R. Tempo, "Randomized algorithms for stability and robustness analysis of high speed communication networks," *IEEE Trans. Neural Networks*, vol. 16, pp. 1229–1241, 2005.

[40] W. Ma, M. Sznaier, and C. M. Lagoa, "A risk adjusted approach to robust simultaneous fault detection and isolation," *Automatica*, vol. 43, no. 3, pp. 499–504, 2007.

[41] R. Rockafellar, *Convex Analysis*. Princeton, NJ: Princeton Univ. Press, 1970.

[42] V. Bogachev, *Measure Theory. Vol. 2*. New York: Springer, 2000.

[43] W. Rudin, *Real & Complex Analysis*. New York: McGraw-Hill, 1987.

[44] M. B. Stinchcombe and H. White, "Some measurability results for extrema of random functions over random sets," *Rev. Economic Studies*, vol. 59, pp. 495–512, 1992.

Sergio Grammatico was born in Marsala, Italy, in 1987. He received the B.Sc. degree in Computer Engineering and the M.Sc. and Ph.D. degrees in Automation Engineering from the University of Pisa, Pisa, Italy, in 2008, 2009, and 2013, respectively, and the M.Sc. degree in Engineering Science from the Sant'Anna School of Advanced Studies, Pisa, Italy, in 2011.

He visited the Department of Mathematics, University of Hawai'i at Manoa in 2010 and 2011, and the Department of Electrical and Computer Engineering, U.C. Santa Barbara in 2012. From 2013 to 2015 he was a Postdoctoral Research Fellow in the Automatic Control Laboratory, ETH Zurich, Switzerland. He is currently an Assistant Professor in the Control Systems group at the Department of Electrical Engineering, Eindhoven University of Technology, The Netherlands. His research interests include Lyapunov control systems, stochastic, game-theoretic control and optimization, with application to large-scale smart energy systems.

Dr. Grammatico was nominated as IEEE TAC Outstanding Reviewer in 2013 and 2014.

Xiaojing Zhang (M'12) received the B.Sc. and M.Sc. degrees in automatic control from the Department of Information Technology and Electrical Engineering, ETH Zurich, Switzerland, in 2010 and 2012, respectively, and is currently pursuing the Ph.D. degree at the Automatic Control Laboratory, ETH Zurich, Switzerland.

In 2011, he spent an academic exchange at the Department of Electrical Engineering and Computer Sciences, U.C. Berkeley. His research interests include stochastic and robust predictive control, randomized algorithms as well as stochastic optimization, with application to energy-efficient building climate control, smart grids, and electric power systems.

Kostas Margellos (S'09–M'14) received the M.S. degree in electrical and computer engineering from the University of Patras, Greece in 2008, and the Ph.D. degree in automatic control from the Department of Information Technology and Electrical Engineering, ETH Zurich, Switzerland, in 2012.

He spent 2013 and 2014 as a Post-Doctoral Researcher in the Automatic Control Laboratory, ETH Zurich and in the Department of Industrial Engineering and Operations Research, UC Berkeley, respectively. As of February 2015, he continues his post-doctoral research at the Polytechnic University of Milan, Milan, Italy. His research interests include optimization and control of complex uncertain systems, with applications to generation and load side control for power networks.

Paul Goulart (M'03) received the B.Sc. and M.Sc. degrees in aeronautics and astronautics from the Massachusetts Institute of Technology (MIT), Cambridge, and the Ph.D. degree in control engineering from the University of Cambridge, Cambridge, U.K., in 2007

He was selected as a Gates Scholar at the University of Cambridge. From 2007 to 2011 he was a Lecturer in control systems in the Department of Aeronautics, Imperial College London, London, U.K., and from 2011 to 2014 a Senior Researcher in the Automatic Control Laboratory, ETH Zurich, Zurich, Switzerland. He is currently an Associate Professor in the Department of Engineering Science and Tutorial Fellow of St. Edmund Hall, University of Oxford, Oxford, U.K. His research interests include model predictive control, robust optimization and control of fluid flows.

John Lygeros (F'11) received the B.Eng. degree in electrical engineering and the M.Sc. degree in systems and control from the Imperial College of Science Technology and Medicine, London, U.K., in 1990 and 1991, respectively, and the Ph.D. degree from the Electrical Engineering and Computer Sciences Department, University of California, Berkeley, in 1996.

He holds the Chair of Computation and Control at the Swiss Federal Institute of Technology (ETH) Zurich, Switzerland, where he is currently serving as the Head of the Automatic Control Laboratory. During the period 1996–2000 he held a series of research appointments at the National Automated Highway Systems Consortium, Berkeley, the Laboratory for Computer Science, M.I.T., and the Electrical Engineering and Computer Sciences Department at U.C. Berkeley. Between 2000 and 2003 he was a University Lecturer at the Department of Engineering, University of Cambridge, U.K., and a Fellow of Churchill College. Between 2003 and 2006 he was an Assistant Professor at the Department of Electrical and Computer Engineering, University of Patras, Greece. In July 2006 he joined the Automatic Control Laboratory at ETH Zurich, first as an Associate Professor, and since January 2010 as Full Professor. His research interests include modeling, analysis, and control of hierarchical, hybrid, and stochastic systems, with applications to biochemical networks, automated highway systems, air traffic management, power grids and camera networks.

Dr. Lygeros is a member of the IET and of the Technical Chamber of Greece.