Hamilton–Jacobi Formulation for Reach–Avoid Differential Games

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***Abstract*—A new framework for formulating reachability problems with competing inputs, nonlinear dynamics, and state constraints as optimal control problems is developed. Such reach–avoid problems arise in, among others, the study of safety problems in hybrid systems. Earlier approaches to reach–avoid computations are either restricted to linear systems, or face numerical difficulties due to possible discontinuities in the Hamiltonian of the optimal control problem. The main advantage of the approach proposed in this paper is that it can be applied to a general class of target-hitting continuous dynamic games with nonlinear dynamics, and has very good properties in terms of its numerical solution, since the value function and the Hamiltonian of the system are both continuous. The performance of the proposed method is demonstrated by applying it to a case study, which involves the target-hitting problem of an underactuated underwater vehicle in the presence of obstacles.**

***Index Terms*—Differential game theory, Hamilton–Jacobi equations, hybrid systems, optimal control, reachability.**

I. Introduction

REACHABILITY for continuous and hybrid systems has been an important topic of research in the dynamics and control literature. Numerous problems regarding safety of air traffic management systems [1]–[3], flight control [4]–[7] ground transportation systems [8], [9], etc., have been formulated in the framework of reachability theory. In most of these applications, the main aim was to design suitable controllers to steer or keep the state of the system in a "safe" part of the state space. The synthesis of such safe controllers for hybrid systems relies on the ability to solve target problems for the case where state constraints are also present. The sets that represent the solution to those problems are known as capture basins [10]. One direct way of computing these sets was proposed in [11] and [12] and was formulated in the context of viability theory [10]. Following the same approach, the authors of [13] and [14] formulated viability, invariance, and pursuit-evasion gaming problems for hybrid systems and used nonsmooth analysis tools to characterize their solutions. Computational tools to support this approach have been developed by [15].

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An alternative, indirect way of characterizing such problems is through the level sets of the value function of an appropriate optimal control problem. By using dynamic programming, for reachability/invariant/viability problems without state constraints, the value function can be characterized as the viscosity solution to a first-order partial differential equation in the standard Hamilton–Jacobi form [16]–[18]. Numerical algorithms based on level set methods have been developed by [19] and [20], have been coded in efficient computational tools by [18] and [21], and can be directly applied to reachability computations.

In the case where state constraints are also present, this target-hitting problem is the solution to a reach–avoid problem in the sense of [1]. The authors of [1] and [22] developed a reach–avoid computation, whose value function was characterized as a solution to a pair of coupled partial differential equations. In [21], [23], and [24], the authors proposed another characterization, which involved only one Hamilton–Jacobi-type partial differential equation together with an inequality constraint. These methods, however, are hampered both from a theoretical and a numerical point of view by the fact that the Hamiltonian of the system is in general discontinuous [22]. In this case, there is no theoretical characterization of the value functions as viscosity solutions of variational equations/inequalities.

In [25] and [26], a scheme based on ellipsoidal techniques to compute reachable sets for control systems with constraints on the state was proposed. This approach was restricted to the class of linear systems. In [27], this approach was extended to a list of interesting target problems with state constraints. The calculation of a solution to the equations proposed in [25]–[27] is in general not easy apart from the case of linear systems, where duality techniques of convex analysis can be used.

In this paper, we propose a new framework of characterizing reach–avoid sets of nonlinear control systems as the solution to an optimal control problem. Related problems in the absence of competing inputs have recently been treated in [28]. We consider the case where we have competing inputs and hence adopt the gaming formulation proposed in [17]. We first restrict our attention to a specific reach–avoid scenario, where the objective of the control input is to make the states of the system hit the target at the end of the time horizon and without violating the state constraints. We then generalize our approach to the case where the controller aims to steer the system toward the target not necessarily at the terminal, but at some time within the specified time horizon. Both problems could be treated as pursuit–evasion games. The contribution of this paper is that it provides a clear characterization of two nonlinear reach–avoid problems, and a proof that the corresponding reach–avoid sets are determined by the level sets of nonsmooth value functions similar to [27], which in turn are the unique continuous viscosity solutions to variational equations of a form similar to [29] and [30]. In addition to theoretical support for the use of computational tools, the numerical advantage of this approach is that the properties of the value function and the Hamiltonian (both of them are continuous) enable the use of existing tools based on Level Set Methods [23], or other tools for solving variational equations [29], to compute the solution of the problem numerically. Another advantage of this paper is that it provides a theoretically solid formulation for the reach–avoid operator, which is the core of the hybrid algorithm of [1] and [22], and consists an alternative approach for the viability-based algorithm of [14].

To illustrate our approach, we consider the motion of an autonomous underwater vehicle in the presence of a disturbance current, whose mathematical modeling was studied in detail in [31]. The objective in this case is to determine the set of initial states from which, for any disturbance, the underwater vehicle can hit a target set while avoiding some fixed obstacles.

In Section II, we pose two reach–avoid problems for continuous systems with competing inputs and state constraints and formulate them in the optimal control framework. Section III provides the characterization of the value functions of these problems as the viscosity solution to two variational equations. In Section IV, we present an application of this approach to the navigation of an underactuated underwater vehicle in the presence of obstacles. Finally, in Section V, we provide some concluding remarks and directions for future work.

II. Differential Games and Reach–Avoid Problems

A. Differential Game Problem Formulation

Consider the continuous time control system $\dot{x} = f(x,u,v)$, and an arbitrary time horizon $T \geq 0$, with $x \in \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}^m$, $v \in V \subseteq \mathbb{R}^p$, and $f(\cdot,\cdot,\cdot) : \mathbb{R}^n \times U \times V \to \mathbb{R}^n$. Let $\mathcal{U}_{[t,t']}$, $\mathcal{V}_{[t,t']}$ denote the set of Lebesgue measurable functions from the interval $[t,t']$ to U, and V respectively. Consider also two bounded, Lipschitz continuous functions $l(\cdot) : \mathbb{R}^n \to \mathbb{R}$, $h(\cdot) : \mathbb{R}^n \to \mathbb{R}$ to be used to encode the target and state constraints respectively.

Assumption 1: The sets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^p$ are compact. The functions $f(x,u,v)$, $l(x)$ and $h(x)$ are bounded, Lipschitz continuous in x, and continuous in u and v.

Under Assumption 1, the system admits a unique solution $x(\cdot) : [t,T] \to \mathbb{R}^n$ for all $t \in [0,T]$, $u(\cdot) \in \mathcal{U}_{[t,T]}$, $v(\cdot) \in \mathcal{V}_{[t,T]}$ and for each initial state $x(t) = x$. For $\tau \in [t,T]$, this solution will be denoted by

$$\phi(\tau, t, x, u(\cdot), v(\cdot)) = x(\tau). \tag{1}$$

Let $C_f > 0$ be a bound such that for all $x, \hat{x} \in \mathbb{R}^n$, $u(\cdot) \in \mathcal{U}_{[t,T]}$ and $v(\cdot) \in \mathcal{V}_{[t,T]}$, and for all $u \in U$ and $v \in V$

$$|f(x,u,v)| \leq C_f \quad |f(x,u,v) - f(\hat{x},u,v)| \leq C_f|x - \hat{x}|.$$

Let also $C_l > 0$ and $C_h > 0$ be such that

$$|l(x)| \leq C_l \quad \text{and} \quad |l(x) - l(\hat{x})| \leq C_l|x - \hat{x}|$$
$$|h(x)| \leq C_h \quad \text{and} \quad |h(x) - h(\hat{x})| \leq C_h|x - \hat{x}|.$$

In a game setting, it is essential to define the information patterns that the two players use. Following [17] and [32], we restrict the first player to play nonanticipative strategies. A nonanticipative strategy is a function $\gamma : \mathcal{V}_{[0,T]} \to \mathcal{U}_{[0,T]}$ such that for all $s \in [t,T]$ and for all $v, \hat{v} \in \mathcal{V}$, if $v(\tau) = \hat{v}(\tau)$ for almost every $\tau \in [t,s]$, then $\gamma[v](\tau) = \gamma[\hat{v}](\tau)$ for almost every $\tau \in [t,s]$. We then use $\Gamma_{[t,T]}$ to denote the class of nonanticipative strategies.

Consider the sets R, A related to the level sets of $l(\cdot) : \mathbb{R}^n \to \mathbb{R}$ and $h(\cdot) : \mathbb{R}^n \to \mathbb{R}$, respectively. For technical purposes, assume that R is closed, whereas A is open. Then, R and A could be characterized as

$$R = \{x \in \mathbb{R}^n | l(x) \leq 0\} \quad A = \{x \in \mathbb{R}^n | h(x) > 0\}.$$

B. Reach–Avoid at the Terminal Time

Let $R \subseteq \mathbb{R}^n$ represent a set that we would like to reach while avoiding a set $A \subseteq \mathbb{R}^n$. One would like to characterize the set of the initial states from which trajectories can start and reach the set R at the terminal time T without passing through the set A over the time horizon $[t,T]$. To answer this question, one needs to determine whether there exists a choice of $\gamma \in \Gamma_{[t,T]}$ such that for all $v(\cdot) \in \mathcal{V}_{[t,T]}$, the trajectory $x(\cdot)$ satisfies $x(T) \in R$ and $x(\tau) \in A^c$ for all $\tau \in [t,T]$. The set of initial conditions that have this property is then

$$\begin{aligned} RA(t,R,A) = \big\{ & x \in \mathbb{R}^n | \exists \gamma(\cdot) \in \Gamma_{[t,T]}, \forall v(\cdot) \in \mathcal{V}_{[t,T]}, \\ & (\phi(T,t,x,\gamma(\cdot),v(\cdot)) \in R) \\ & \wedge (\forall \tau \in [t,T], \phi(\tau,t,x,\gamma(\cdot),v(\cdot)) \notin A) \big\}. \end{aligned} \tag{2}$$

Now, introduce the value function $V : \mathbb{R}^n \times [0,T] \to \mathbb{R}$

$$\begin{aligned} V(x,t) = \inf_{\gamma(\cdot)\in\Gamma_{[t,T]}} \sup_{v(\cdot)\in\mathcal{V}_{[t,T]}} \max \Big\{ & l(\phi(T,t,x,\gamma(\cdot),v(\cdot))), \\ & \max_{\tau\in[t,T]} h(\phi(\tau,t,x,\gamma(\cdot),v(\cdot))) \Big\}. \end{aligned} \tag{3}$$

V[1] can be thought of as the value function of a differential game, where u is trying to minimize, whereas v is trying to maximize the maximum between the value attained by l at the end T of the time horizon and the maximum value attained by h along the state trajectory over the horizon $[t,T]$. Based on [16], [17], and [29], we will show that the value function defined by (3) is the unique viscosity solution of the following variational equation:

$$\begin{aligned} \max \Big\{ & h(x) - V(x,t), \frac{\partial V}{\partial t}(x,t) \\ & + \sup_{v\in V} \inf_{u\in U} \frac{\partial V}{\partial x}(x,t) f(x,u,v) \Big\} = 0 \end{aligned} \tag{4}$$

with terminal condition $V(x,T) = \max\{l(x), h(x)\}$. It is then easy to link the set $RA(t,R,A)$ of (2) to the level set of the value function $V(x,t)$ defined in (3).

Proposition 1: $RA(t,R,A) = \{x \in \mathbb{R}^n | V(x,t) \leq 0\}$.

Proof: We first show that $RA(t,R,A) \subseteq \{x \in \mathbb{R}^n | V(x,t) \leq 0\}$ holds. Consider a point $x \in RA(t,R,A)$, and for the sake of contradiction assume that $V(x,t) > 0$. The latter implies that $\exists \epsilon > 0$ such that $\inf_{\gamma(\cdot)\in\Gamma_{[t,T]}} \sup_{v(\cdot)\in\mathcal{V}_{[t,T]}} \max\{l(\phi(T,t,x,\gamma(\cdot),v(\cdot))), \max_{\tau\in[t,T]} h(\phi(\tau,t,x,\gamma(\cdot),v(\cdot)))\} > 2\epsilon > 0$. Equivalently, there exists $\epsilon > 0$, such that for all $\gamma(\cdot) \in \Gamma_{[t,T]}$,

[1]Note that this V is different from the set that v takes values from, which was defined in Section II-A. Throughout the paper, it will always be clear from the context to which V we refer.

there exists $v(\cdot) \in \mathcal{V}_{[t,T]}$, so that $\max\{l(\phi(T,t,x,\gamma(\cdot), v(\cdot))), \max_{\tau\in[t,T]} h(\phi(\tau,t,x,\gamma(\cdot),v(\cdot)))\} > \epsilon > 0$. The last statement is equivalent to there exists $\epsilon > 0$, such that for all $\gamma(\cdot) \in \Gamma_{[t,T]}$, there exists $v(\cdot) \in \mathcal{V}_{[t,T]}$, such that $l(\phi(T,t,x,\gamma(\cdot),v(\cdot))) > \epsilon > 0$ or there exists $\tau \in [t,T]$ such that $h(\phi(\tau,t,x,\gamma(\cdot),v(\cdot))) > \epsilon > 0$. Or in other words, there exists $\epsilon > 0$, such that for all $\gamma(\cdot) \in \Gamma_{[t,T]}$, there exists $v(\cdot) \in \mathcal{V}_{[t,T]}$, so that $\phi(T,t,x,\gamma(\cdot),v(\cdot)) \notin R$ or there exists $\tau \in [t,T]\phi(\tau,t,x,\gamma(\cdot),v(\cdot)) \in A$. The last statement is equivalent to $x \notin RA(t,R,A)$, which is a contradiction.

We now show that $\{x \in \mathbb{R}^n | V(x,t) \leq 0\} \subseteq RA(t,R,A)$. Consider (x,t) such that $V(x,t) \leq 0$, and for the sake of contradiction assume that $x \notin RA(t,R,A)$. This implies that for all $\gamma(\cdot) \in \Gamma_{[t,T]}$, there exists $v(\cdot) \in \mathcal{V}_{[t,T]}$, such that $\phi(T,t,x,\gamma(\cdot),v(\cdot)) \notin R$ or there exists $\tau^* \in [t,T]$ such that $\phi(\tau^*,t,x,\gamma(\cdot),v(\cdot)) \in A$. Then, there exists $\delta > 0$ such that for all $\gamma(\cdot) \in \Gamma_{[t,T]}$, there exists $v(\cdot) \in \mathcal{V}_{[t,T]}$, such that $l(\phi(T,t,x,\gamma(\cdot),v(\cdot))) > \delta > 0$, or there exists $\tau^* \in [t,T]$ such that $h(\phi(\tau^*,t,x,\gamma(\cdot),v(\cdot))) > \delta > 0$. However, $V(x,t) \leq 0$ implies that for all $\epsilon > 0$ there exists a strategy $\gamma(\cdot) \in \Gamma_{[t,T]}$ such that $\sup_{v(\cdot)\in\mathcal{V}_{[t,T]}} \max\{l(\phi(T,t,x,\gamma(\cdot),v(\cdot))), \max_{\tau\in[t,T]} h(\phi(\tau,t,x,\gamma(\cdot),v(\cdot)))\} \leq \epsilon$. Hence for all $v(\cdot) \in \mathcal{V}_{[t,T]}$, $l(\phi(T,t,x,\gamma(\cdot),v(\cdot))) \leq \epsilon$ and also for all $\tau \in [t,T]$, $h(\phi(\tau,t,x,\gamma(\cdot),v(\cdot))) \leq \epsilon$. The last argument implies that $l(\phi(T,x,t,\gamma(\cdot),v(\cdot))) \leq \epsilon$, and for all $\tau \in [t,T]$, and so also for $\tau = \tau^*$, $h(\phi(\tau^*,x,t,\gamma(\cdot),v(\cdot))) \leq \epsilon$. By choosing $\epsilon = \delta/2$, the last statement establishes a contradiction and completes the proof. ■

C. Reach–Avoid at Any Time

Another related problem that one might need to characterize is the set of initial states from which trajectories can start, and for any disturbance input can reach the set R not at the terminal, but at some time within the time horizon $[t,T]$, and without passing through the set A until they hit R. In other words, we would like to determine the set

$$\begin{aligned}\widetilde{RA}(t,R,A) &= \{x \in \mathbb{R}^n | \exists\gamma(\cdot) \in \Gamma_{[t,T]}, \forall v(\cdot) \in \mathcal{V}_{[t,T]}, \exists\tau_1 \in [t,T], \\ &\quad (\phi(\tau_1,t,x,\gamma(\cdot),v(\cdot)) \in R) \\ &\quad \wedge (\forall\tau_2 \in [t,\tau_1], \phi(\tau_2,t,x,\gamma(\cdot),v(\cdot)) \notin A)\}. \end{aligned} \tag{5}$$

Based on [33], define the augmented input as $\tilde{u} = (u,\bar{u}) \in U \times [0,1]$ and consider the dynamics

$$\tilde{f}(x,\tilde{u},v) = \bar{u}f(x,u,v). \tag{6}$$

In Assumption 1, $f(x,u,v)$ is assumed to be continuous in u and v, and Lipschitz continuous in x. Hence, since $\tilde{u}$ is the augmented input, $\bar{u}$ is not binary but takes values in [0, 1], and $\tilde{f}(x,\tilde{u},v)$ is affine in $\bar{u}$, if $f(x,u,v)$ satisfies Assumption 1 so will $\tilde{f}(x,\tilde{u},v)$. Let $\tilde{\phi}(\tau,x,t,\tilde{u}(\cdot),v(\cdot))$ denote the solution of the augmented system, and define $\widetilde{U}$, $\widetilde{\mathcal{U}}$ and $\widetilde{\Gamma}$ similarly to the previous case. Following [33], for every $\tilde{u} \in \widetilde{\mathcal{U}}_{[t,T]}$, the pseudo-time variable $\sigma : [t,T] \to [t,T]$ is given by

$$\sigma(\tau) = t + \int_t^\tau \bar{u}(s)ds. \tag{7}$$

Consider σ^*, as it was defined in [33], such that $\sigma(\sigma^*(\tau)) = \tau$. In [33, Lemma 6], σ^* was proven to be the limit of a convergent sequence of functions, its existence was verified, and it was shown that

$$\phi(\sigma(\tau),x,t,u(\sigma^*(\cdot)),v(\sigma^*(\cdot))) = \tilde{\phi}(\tau,x,t,\tilde{u}(\cdot),v(\cdot)) \tag{8}$$

for any $\tau \in [t,T]$. Based on the analysis of [33], (8) implies that the trajectory $\tilde{\phi}$ of the augmented system visits only the subset of the states visited by the trajectory ϕ of the original system in the time interval $[t,\sigma(\tau)]$.

Define now the value function

$$\begin{aligned}\widetilde{V}(x,t) = \inf_{\tilde{\gamma}(\cdot)\in\widetilde{\Gamma}_{[t,T]}} \sup_{v(\cdot)\in\mathcal{V}_{[t,T]}} \max\Big\{ & l\Big(\tilde{\phi}(T,t,x,\tilde{\gamma}[v](\cdot),v(\cdot))\Big), \\ & \max_{\tau\in[t,T]} h\Big(\tilde{\phi}(\tau,t,x,\tilde{\gamma}[v](\cdot),v(\cdot))\Big)\Big\}.\end{aligned}$$

One can then show that $\widetilde{V}$ is related to the set $\widetilde{RA}$.

Proposition 2: For $\tau \in [0,T]$, $\widetilde{RA}(\tau,R,A) = \{x \in \mathbb{R}^n | \widetilde{V}(x,\tau) \leq 0\}$.

The proof of this proposition is given in Appendix A.

III. Characterization of the Value Function

A. Basic Properties of V

We first establish the consequences of the principle of optimality for V.

Lemma 1: For all $(x,t) \in \mathbb{R}^n \times [0,T]$ and all $\alpha \in [0,T-t]$, we have (9), shown at the bottom of the page. Moreover, for all $(x,t) \in \mathbb{R}^n \times [0,T], V(x,t) \geq h(x)$.

The proof for the second part is straightforward and follows from the definition of V. The proof for the first part is given in Appendix B.

Next, we show that V is a bounded, Lipschitz continuous function.

Lemma 2: There exists a constant $C > 0$ such that for all $(x,t),(\hat{x},\hat{t}) \in \mathbb{R}^n \times [0,T]$

$$\begin{aligned}|V(x,t)| &\leq C \\ \left|V(x,t) - V(\hat{x},\hat{t})\right| &\leq C\left(|x-\hat{x}| + |t-\hat{t}|\right).\end{aligned}$$

The proof of this Lemma is given in Appendix B.

$$V(x,t) = \inf_{\gamma(\cdot)\in\Gamma_{[t,t+\alpha]}} \sup_{v(\cdot)\in\mathcal{V}_{[t,t+\alpha]}} \left[\max\left\{\max_{\tau\in[t,t+\alpha]} h(\phi(\tau,t,x,\gamma(\cdot),v(\cdot))), V(\phi(t+\alpha,t,x,\gamma(\cdot),v(\cdot)),t+\alpha)\right\}\right]. \tag{9}$$

B. Variational Equation for V

We now introduce the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, defined by

$$H(p,x) = \sup_{v\in V} \inf_{u\in U} p^T f(x,u,v).$$

Lemma 3: There exists a constant $C > 0$ such that for all $p, q \in \mathbb{R}^n$, and all $x, y \in \mathbb{R}^n$

$$|H(p,x) - H(q,x)| < C|p-q|$$
$$|H(p,x) - H(p,y)| < C|p||x-y|.$$

The proof of this fact is straightforward (see [16] or [34, Lemma 2]). We are now in a position to state and prove the following theorem, which is the main result of this section.

Theorem 1: The function V is the unique viscosity solution over $(x,t) \in \mathbb{R}^n \times [0,T]$ of the variational equation

$$\max\left\{h(x) - V(x,t), \frac{\partial V}{\partial t}(x,t) + \sup_{v\in V}\inf_{u\in U} \frac{\partial V}{\partial x}(x,t) f(x,u,v)\right\} = 0$$

with terminal condition $V(x,T) = \max\{l(x), h(x)\}$.

Proof: Uniqueness follows from [29, Lemmas 2 & 3 and Proposition 1]. Note also that $V(x,T) = \max\{l(x), h(x)\}$ by definition of the value function. Therefore, it suffices to show the following.

1) For all $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and for all smooth $W : \mathbb{R}^n \to \mathbb{R}$, if $V - W$ attains a local maximum at (x_0,t_0), then

$$\max\left\{h(x_0) - V(x_0,t_0), \frac{\partial W}{\partial t}(x_0,t_0) + \sup_{v\in V}\inf_{u\in U} \frac{\partial W}{\partial x}(x_0,t_0) f(x_0,u,v)\right\} \geq 0.$$

2) For all $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and for all smooth $W : \mathbb{R}^n \to \mathbb{R}$, if $V - W$ attains a local minimum at (x_0,t_0), then

$$\max\left\{h(x_0) - V(x_0,t_0), \frac{\partial W}{\partial t}(x_0,t_0) + \sup_{v\in V}\inf_{u\in U} \frac{\partial W}{\partial x}(x_0,t_0) f(x_0,u,v)\right\} \leq 0.$$

The case $t = 0$ is automatically captured by [35, p. 546].

Part 1: Consider an arbitrary $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and a smooth $W : \mathbb{R}^n \times (0,T) \to \mathbb{R}$ such that $V - W$ has a local maximum at (x_0,t_0). Then, there exists $\delta_1 > 0$ such that for all $(x,t) \in \mathbb{R}^n \times (0,T)$ with $|x-x_0|^2 + (t-t_0)^2 < \delta_1$

$$(V-W)(x_0,t_0) \geq (V-W)(x,t).$$

We would like to show that

$$\max\left\{h(x_0) - V(x_0,t_0), \frac{\partial W}{\partial t}(x_0,t_0) + \sup_{v\in V}\inf_{u\in U} \frac{\partial W}{\partial x}(x_0,t_0) f(x_0,u,v)\right\} \geq 0.$$

Since by Lemma 1 $h(x) - V(x,t) \leq 0$, either $h(x_0) = V(x_0,t_0)$ or, $h(x_0) - V(x_0,t_0) < 0$. For the former, the claim holds, whereas for the latter, it suffices to show that there exists $v \in V$ such that for all $u \in U$

$$\frac{\partial W}{\partial t}(x_0,t_0) + \frac{\partial W}{\partial x}(x_0,t_0) f(x_0,u,v) \geq 0.$$

For the sake of contradiction, assume that for all $v \in V$, there exists $u \in U$ such that for some $\theta > 0$

$$\frac{\partial W}{\partial t}(x_0,t_0) + \frac{\partial W}{\partial x}(x_0,t_0) f(x_0,u,v) < -2\theta < 0.$$

Since W is smooth and f is continuous, then based on [17], we have that

$$\frac{\partial W}{\partial t}(x_0,t_0) + \frac{\partial W}{\partial x}(x_0,t_0) f(x_0,u,\zeta) < -\frac{3\theta}{2} < 0$$

for all $\zeta \in B(v,r) \cap V$ and some $r > 0$, where $B(v,r)$ denotes a ball centered at v with radius r. Because V is compact, there exist finitely many distinct points $v_1,\ldots,v_n \in V, u_1,\ldots,u_n \in U$, and $r_1,\ldots,r_n > 0$ such that $V \subset \bigcup_{i=1}^n B(v_i,r_i)$ and for $\zeta \in B(v_i,r_i)$

$$\frac{\partial W}{\partial t}(x_0,t_0) + \frac{\partial W}{\partial x}(x_0,t_0) f(x_0,u_i,\zeta) < -\frac{3\theta}{2} < 0.$$

Define $g : V \to U$ by setting for $k = 1,\ldots,n$, $g(v) = u_k$ if $v \in B(u_k,r_k) \backslash \bigcup_{i=1}^{k-1} B(u_i,r_i)$. Then

$$\frac{\partial W}{\partial t}(x_0,t_0) + \frac{\partial W}{\partial x}(x_0,t_0) f\left(x_0,g(v),v\right) < -\frac{3\theta}{2} < 0.$$

Since W is smooth and f is continuous, there exists $\delta_2 \in (0,\delta_1)$ such that for all $(x,t) \in \mathbb{R}^n \times (0,T)$ with $|x-x_0|^2 + (t-t_0)^2 < \delta_2$

$$\frac{\partial W}{\partial t}(x,t) + \frac{\partial W}{\partial x}(x,t) f\left(x,g(v),v\right) < -\theta < 0.$$

Finally, define $\gamma : \mathcal{V}_{[t_0,T]} \to \mathcal{U}_{[t_0,T]}$ by $\gamma[v](\tau) = g(v(\tau))$ for all $\tau \in [t_0,T]$. It is easy to see that γ is now nonanticipative and hence $\gamma(\cdot) \in \Gamma_{[t_0,T]}$. Thus for all $v(\cdot) \in V_{[t_0,T]}$ and all $(x,t) \in \mathbb{R}^n \times (0,T)$ such that $|x-x_0|^2 + (t-t_0)^2 < \delta_2$

$$\frac{\partial W}{\partial t}(x,t) + \frac{\partial W}{\partial x}(x,t) f\left(x,\gamma[v](\cdot),v(\cdot)\right) < -\theta < 0.$$

By continuity, there exists $\delta_3 > 0$ such that $|\phi(t,t_0,x_0,\gamma[v](\cdot),v(\cdot)) - x_0|^2 + (t-t_0)^2 < \delta_2$ for all $t \in [t_0, t_0+\delta_3]$. Therefore, for all $v(\cdot) \in V_{[t_0,T]}$

$$\begin{aligned} &V(\phi(t,t_0,x_0,\gamma(\cdot),v(\cdot)),t) - V(x_0,t_0)\\ &\quad\le W(\phi(t,t_0,x_0,\gamma(\cdot),v(\cdot)),t) - W(x_0,t_0)\\ &\quad= \int_{t_0}^{t} \left(\frac{\partial W}{\partial s}(\phi(s,t_0,x_0,\gamma(\cdot),v(\cdot)),s)\right.\\ &\qquad+\frac{\partial W}{\partial x}(\phi(s,t_0,x_0,\gamma(\cdot),v(\cdot)),s)\\ &\qquad\left.\times f(\phi(s,t_0,x_0,\gamma(\cdot),v(\cdot)),\gamma(\cdot),v(\cdot))\right)ds\\ &\quad< -\theta(t-t_0). \end{aligned}$$

Let $\tau_0 \in [t_0, t_0+\delta_3]$ be such that $h(\phi(\tau_0,t_0,x_0,\gamma(\cdot),v(\cdot)))= \max_{\tau\in[t_0,t_0+\delta_3]} h(\phi(\tau,t_0,x_0,\gamma(\cdot),v(\cdot)))$.

Case 1.1: If $\tau_0 \in (t_0, t_0+\delta_3]$, then for $t=\tau_0$ we have

$$V(\phi(\tau_0,t_0,x_0,\gamma(\cdot),v(\cdot)),\tau_0) - V(x_0,t_0) < -\theta(\tau_0 - t_0) < 0. \quad (10)$$

Then, by the dynamic programming argument of Lemma 1, we have

$$\begin{aligned} &V(x_0,t_0)\\ &\le \sup_{v(\cdot)\in\mathcal{V}_{[t_0,t_0+\delta_3]}} \left[\max\left\{\max_{\tau\in[t_0,t_0+\delta_3]} h(\phi(\tau,t_0,x_0,\gamma(\cdot),v(\cdot))),\right.\right.\\ &\qquad\left.\left. V(\phi(\tau_0,t_0,x_0,\gamma(\cdot),v(\cdot)),\tau_0)\right\}\right]. \end{aligned}$$

We can choose $\hat{v}(\cdot) \in \mathcal{V}_{[t_0,t_0+\delta_3]}$ such that

$$\begin{aligned} V(x_0,t_0) \le \max&\left\{\max_{\tau\in[t_0,t_0+\delta_3]} h(\phi(\tau,t_0,x_0,\gamma(\cdot),v(\cdot))),\right.\\ &\left. V(\phi(\tau_0,t_0,x_0,\gamma(\cdot),v(\cdot)),\tau_0)\right\} + \epsilon \end{aligned}$$

and set $\epsilon < (\theta/2)(\tau_0 - t_0)$. Since by Lemma 1 $h(x) - V(x,t) \le 0$ for all $(x,t) \in \mathbb{R}^n \times (0,T)$, we have that $\max_{\tau\in[t_0,t_0+\delta_3]} h(\phi(\tau,t_0,x_0,\gamma(\cdot),\hat{v}(\cdot)))= h(\phi(\tau_0,t_0,x_0,\gamma(\cdot),\hat{v}(\cdot)))\le V(\phi(\tau_0,t_0,x_0,\gamma(\cdot),\hat{v}(\cdot)),\tau_0)$. Hence

$$V(x_0,t_0) \le V(\phi(\tau_0,t_0,x_0,\gamma(\cdot),\hat{v}(\cdot)),\tau_0) + \frac{\theta}{2}(\tau_0 - t_0).$$

Since (10) holds for all $v(\cdot) \in V_{[t_0,T]}$, it will also hold for $\hat{v}(\cdot)$, and hence the last argument establishes a contradiction.

Case 1.2: If $\tau_0 = t_0$ then for $t = t_0+\delta_3$ we have that for all $v(\cdot) \in V_{[t_0,T]}$

$$V(\phi(t_0+\delta_3,t_0,x_0,\gamma(\cdot),v(\cdot)),t_0+\delta_3) - V(x_0,t_0) < -\theta\delta_3 < 0.$$

Since by Lemma 1

$$\begin{aligned} V(x_0,t_0) \le \sup_{v(\cdot)\in\mathcal{V}_{[t_0,t_0+\delta_3]}} \max&\left\{\max_{\tau\in[t_0,t_0+\delta_3]} h(\phi(\tau,t_0,x_0,\right.\\ &\left.\gamma(\cdot),v(\cdot))), V(\phi(t_0+\delta_3,t_0,x_0,\gamma(\cdot),v(\cdot)),t_0+\delta_3)\right\} \end{aligned}$$

then if

$$\begin{aligned} V(x_0,t_0) \le \sup_{v(\cdot)\in\mathcal{V}_{[t_0,t_0+\delta_3]}} &V(\phi(t_0+\delta_3,t_0,\\ &x_0,\gamma(\cdot),v(\cdot)),t_0+\delta_3) \end{aligned}$$

we can choose $\hat{v}(\cdot) \in \mathcal{V}_{[t_0,t_0+\delta_3]}$ such that

$$V(x_0,t_0) \le V(\phi(t_0+\delta_3,t_0,x_0,\gamma(\cdot),\hat{v}(\cdot)),t_0+\delta_3) + \frac{\theta\delta_3}{2}$$

which establishes a contradiction.

If

$$V(x_0,t_0) \le \sup_{v(\cdot)\in\mathcal{V}_{[t_0,t_0+\delta_3]}} \max_{\tau\in[t_0,t_0+\delta_3]} h(\phi(\tau,t_0,x_0,\gamma(\cdot),v(\cdot)))$$

then we can choose $\hat{v}(\cdot) \in \mathcal{V}_{[t_0,t_0+\delta_3]}$ such that

$$V(x_0,t_0) \le \max_{\tau\in[t_0,t_0+\delta_3]} h(\phi(\tau,t_0,x_0,\gamma(\cdot),\hat{v}(\cdot))) + \epsilon$$

or equivalently $V(x_0,t_0) \le h(x_0) + \epsilon$, since $\tau_0 = t_0$. Based on our initial hypothesis that $h(x_0) < V(x_0,t_0)$, there exists a $\delta > 0$ such that $h(x_0) - V(x_0,t_0) < -2\delta$. If we take $\epsilon < \delta$, we establish a contradiction.

Part 2: Consider an arbitrary $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and a smooth $W : \mathbb{R}^n \times (0,T) \to \mathbb{R}$ such that $V - W$ has a local minimum at (x_0,t_0). Then, there exists $\delta_1 > 0$ such that for all $(x,t) \in \mathbb{R}^n \times (0,T)$ with $|x-x_0|^2 + (t-t_0)^2 < \delta_1$

$$(V-W)(x_0,t_0) \le (V-W)(x,t).$$

We would like to show that

$$\begin{aligned} \max&\left\{h(x_0) - V(x_0,t_0), \frac{\partial W}{\partial t}(x_0,t_0)\right.\\ &\left.+\sup_{v\in V}\inf_{u\in U} \frac{\partial W}{\partial x}(x_0,t_0)f(x_0,u,v)\right\} \le 0. \end{aligned}$$

Since we have $V(x,t) \ge h(x)$, it suffices to show that $(\partial W/\partial t)(x_0,t_0) + \sup_{v\in V}\inf_{u\in U}(\partial W/\partial x)(x_0,t_0)f(x_0,u,v) \le 0$. This implies that for all $v \in V$, there exists a $u \in U$ such that

$$\frac{\partial W}{\partial t}(x_0,t_0) + \frac{\partial W}{\partial x}(x_0,t_0)f(x_0,u,v) \le 0.$$

For the sake of contradiction, assume that there exists $\hat{v} \in V$ such that for all $u \in U$ there exists $\theta > 0$ such that

$$\frac{\partial W}{\partial t}(x_0,t_0) + \frac{\partial W}{\partial x}(x_0,t_0)f(x_0,u,\hat{v}) > 2\theta > 0.$$

Since W is smooth, there exists $\delta_2 \in (0,\delta_1)$ such that for all $(x,t) \in \mathbb{R}^n \times (0,T)$ with $|x-x_0|^2 + (t-t_0)^2 < \delta_2$

$$\frac{\partial W}{\partial t}(x,t) + \frac{\partial W}{\partial x}(x,t)f(x,u,\hat{v}) > \theta > 0.$$

Hence, following [17], for $v(\cdot)\equiv\hat{v}$ and any $\gamma(\cdot)\in\Gamma_{[t_0,T]}$

$$\frac{\partial W}{\partial t}(x,t)+\frac{\partial W}{\partial x}(x,t)f\left(x,\gamma(\cdot),v(\cdot)\right)>\theta>0.$$

By continuity, there exists $\delta_3>0$ such that $|\phi(t,t_0,x_0,\gamma(\cdot),v(\cdot))-x_0|^2+(t-t_0)^2<\delta_2$ for all $t\in[t_0,t_0+\delta_3]$. Therefore, for all $\gamma(\cdot)\in\Gamma_{[t_0,T]}$

$$\begin{aligned}
&V\left(\phi\left(t_0+\delta_3,t_0,x_0,\gamma(\cdot),v(\cdot)\right),t_0+\delta_3\right)-V(x_0,t_0)\\
&\quad\ge W\left(\phi\left(t_0+\delta_3,t_0,x_0,\gamma(\cdot),v(\cdot)\right),t_0+\delta_3\right)-W(x_0,t_0)\\
&\quad=\int_{t_0}^{t_0+\delta_3}\left(\frac{\partial W}{\partial t}\left(\phi\left(t,t_0,x_0,\gamma(\cdot),v(\cdot)\right),t\right)\right.\\
&\qquad\qquad+\frac{\partial W}{\partial x}\left(\phi\left(t,t_0,x_0,\gamma(\cdot),v(\cdot)\right),t\right)\\
&\qquad\qquad\left.\times f\left(\phi\left(t,t_0,x_0,\gamma(\cdot),v(\cdot)\right),\gamma(\cdot),v(\cdot)\right)\right)dt\\
&\quad>\theta\delta_3.
\end{aligned}$$

However, by the dynamic programming argument of Lemma 1, we can choose a $\hat{\gamma}(\cdot)\in\Gamma_{[t_0,T]}$ such that we have the equation shown at the bottom of the page. The last statement establishes a contradiction and completes the proof. ■

C. Variational Equation for $\widetilde{V}$

Consider the value function $\widetilde{V}$ defined in the previous section. The following theorem proposes that $\widetilde{V}$ is the unique viscosity solution of another variational equation.

Theorem 2: $\widetilde{V}:\mathbb{R}^n\times[0,T]\to\mathbb{R}$ is the unique viscosity solution of the variational equation

$$\max\left\{h(x)-\widetilde{V}(x,t),\frac{\partial\widetilde{V}}{\partial t}(x,t)+\min\left\{0,\sup_{v\in V}\inf_{u\in U}\frac{\partial\widetilde{V}}{\partial x}(x,t)f(x,u,v)\right\}\right\}=0 \quad (11)$$

with terminal condition $\widetilde{V}(x,T)=\max\{l(x),h(x)\}$.

Proof: By Theorem 1, $\widetilde{V}(x,t)$ is the unique viscosity solution of (4), subject to $\widetilde{V}(x,T)=\max\{l(x),h(x)\}$. If we let $\widetilde{H}(x,p)=\sup_{v\in V}\inf_{\tilde{u}\in\widetilde{U}}p^T\tilde{f}(x,u,v)$, then, following the proof of [33, Theorem 2], we have that

$$\begin{aligned}
\widetilde{H}(x,p)&=\sup_{v\in V}\inf_{\tilde{u}\in\widetilde{U}}p^T\tilde{f}(x,\tilde{u},v)\\
&=\sup_{v\in V}\inf_{u\in U}\inf_{\bar{u}\in\bar{U}}p^T\left(\bar{u}f(x,u,v)\right)\\
&=\inf_{\bar{u}\in\bar{U}}\bar{u}\sup_{v\in V}\inf_{u\in U}p^Tf(x,u,v)\\
&=\min_{\bar{u}\in\bar{U}}\bar{u}\sup_{v\in V}\inf_{u\in U}p^Tf(x,u,v)\\
&=\min\left\{0,H(x,p)\right\}.
\end{aligned}$$

Consequently, the two variational equations (4) and (11) are equivalent, and so $\widetilde{V}(x,t)$ is the viscosity solution of (11). ■

Since the solution to (11) is unique [29], one could easily show that

$$\widetilde{V}(x,t)=\inf_{\gamma(\cdot)\in\Gamma_{[t,T]}}\sup_{v(\cdot)\in\mathcal{V}_{[t,T]}}\min_{\tau_1\in[t,T]}\max\left\{l\left(\phi\left(\tau_1,t,x,\gamma(\cdot),v(\cdot)\right)\right),\max_{\tau_2\in[t,\tau_1]}h\left(\phi\left(\tau_2,t,x,\gamma(\cdot),v(\cdot)\right)\right)\right\}.$$

IV. Case Study: Underwater Vehicle Motion in the Presence of Obstacles

To illustrate the theoretical formulation of Section II, we consider the motion of an underactuated underwater vehicle in the presence of a disturbance current. Based on the modeling approach of [31], we focus on the problem of steering the vehicle toward a specified target set while avoiding fixed obstacles in the navigation space.

A. Mathematical Modeling

Following the detailed derivation of [31], we consider the motion of a three-degrees-of-freedom underwater vehicle with two back thrusters but no side thruster. For the kinematic equations of the vehicle, the following model was assumed:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}\alpha\cos(\beta)\\ \alpha\sin(\beta)\\ 0\end{bmatrix}+\begin{bmatrix}\cos(x_3)\\ \sin(x_3)\\ 0\end{bmatrix}u_1+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u_2+\begin{bmatrix}-\sin(x_3)\\ \cos(x_3)\\ 0\end{bmatrix}v_1.$$

$$\begin{aligned}
V(x_0,t_0)&\ge\sup_{v(\cdot)\in\mathcal{V}_{[t_0,t_0+\delta_3]}}\left[\max\left\{\max_{\tau\in[t_0,t_0+\delta_3]}h\left(\phi\left(\tau,t_0,x_0,\hat{\gamma}(\cdot),v(\cdot)\right)\right),V\left(\phi\left(t_0+\delta_3,t_0,x_0,\hat{\gamma}(\cdot),v(\cdot)\right),t_0+\delta_3\right)\right\}\right]-\frac{\delta_3\theta}{2}\\
&\ge\max\left\{\max_{\tau\in[t_0,t_0+\delta_3]}h\left(\phi\left(\tau,t_0,x_0,\hat{\gamma}(\cdot),v(\cdot)\right)\right),V\left(\phi\left(t_0+\delta_3,t_0,x_0,\hat{\gamma}(\cdot),v(\cdot)\right),t_0+\delta_3\right)\right\}-\frac{\delta_3\theta}{2}\\
&\ge V\left(\phi\left(t_0+\delta_3,t_0,x_0,\hat{\gamma}(\cdot),v(\cdot)\right),t_0+\delta_3\right)-\frac{\delta_3\theta}{2}.
\end{aligned}$$

The state variables in $x = [x_1\, x_2\, x_3]^T$ represent the cartesian coordinates and the orientation of the vehicle, whereas u_1, v_1 are the components of the linear velocity vector, and u_2 is the angular velocity. Variable α denotes the amplitude and β the direction of the disturbance current.

In [31], the dynamic and kinematic equations of motion were studied separately, and a forward reachability analysis for the dynamical subsystem was performed, in order to determine an estimate for the bounds of u_1, u_2 and v_1. By adopting these results, we further consider, as in [31], that $u = [u_1\, u_2]^T$ is the control input, which consists of the velocities along the two degrees of freedom, and v_1 to act as a bounded disturbance, since it is the velocity along the unactuated degree of freedom.

B. Reach–Avoid Formulation

The objective in this problem is to identify *the set of initial states for which there exists a control input* u, *such that for any disturbance* v, *the vehicle can reach a target* R *within some specified time interval, while avoiding some fixed obstacles denoted by* A. This is a reach–avoid at any time problem, and based on the analysis of Section II-B, the value function $\widetilde{V}$, that characterizes the desired set, is the viscosity solution of (11). The target set R is characterized by constraints of the form $x^R_{1,\min} \leq x_1 \leq x^R_{1,\max}$ and $x^R_{2,\min} \leq x_2 \leq x^R_{2,\max}$. Similarly, A represents the obstacles in the motion space and could be expressed as $A = \cup_{i=1}^N A_i$, where $A_i = \{x \in \mathbb{R}^3 | x^{A_i}_{1,\min} \leq x_1 \leq x^{A_i}_{1,\max}$ and $x^{A_i}_{2,\min} \leq x_2 \leq x^{A_i}_{2,\max}\}$, and N denotes the number of obstacles. To encode these constraints in the reach–avoid setting of Section II, we define functions $l(\cdot) : \mathbb{R}^3 \to \mathbb{R}$ and $h(\cdot) : \mathbb{R}^3 \to \mathbb{R}$, such that $l(\cdot)$ characterizes the set R and $h(x) = \max\{h^{A_1}(x), \ldots, h^{A_N}(x)\}$, where $h^{A_i}(x)$ determines the obstacle i. A natural choice is to choose $l(\cdot)$ to be the signed distance to the set R^c. Then

$$l(x) = \begin{cases} d(x,R), & \text{if } x \in R^c \\ -d(x,R^c), & \text{if } x \in R \end{cases}$$

where $d(x,R) = \inf_{\hat{x}\in R} |x - \hat{x}|$ stands for the usual distance to the set R. Similarly, $h^{A_i}(\cdot)$ is defined to be the signed distance to the set A_i respectively. The functions $l(\cdot)$ and $h(\cdot)$ will then be Lipschitz by construction; to keep them bounded, we saturated them at the Lipschitz constants C_l and C_h, respectively.

The Hamiltonian of the system, as defined in Section III-B, is given by

$$H(p,x) = \sup_{v\in V} \inf_{u\in U} \left((p_1 \cos(x_3) + p_2 \sin(x_3))\, u_1 + p_3 u_2 \right.$$
$$\left. + (-p_1 \sin(x_3) + p_2 \cos(x_3))\, v_1 + \alpha p_1 \cos(\beta) + \alpha p_2 \sin(\beta) \right).$$

The input values that optimize $H(p,x)$ can be then easily computed as

$$u_1^* = \begin{cases} u_{1,\min}, & \text{if } p_1\cos(x_3) + p_2 \sin(x_3) \geq 0 \\ u_{1,\max}, & \text{if } p_1\cos(x_3) + p_2 \sin(x_3) < 0 \end{cases}$$
$$u_2^* = \begin{cases} u_{2,\min}, & \text{if } p_3 \geq 0 \\ u_{2,\max}, & \text{if } p_3 < 0 \end{cases}$$
$$v_1^* = \begin{cases} v_{1,\max}, & \text{if } -p_1\sin(x_3) + p_2 \cos(x_3) \geq 0 \\ v_{1,\min}, & \text{if } -p_1\sin(x_3) + p_2 \cos(x_3) < 0 \end{cases}$$

where $p_i = (\partial \widetilde{V}_i / \partial x_i)$ for $i = 1,2,3$ (see [31] for a detailed derivation). Although these inputs depend in general on the state of the system (through the costate vector p), they are not necessarily feedback, but nonanticipative strategies. For a single input setting though, the optimal control inputs would also be feedback.

To enforce the constraints represented by $h(\cdot)$ numerically, a procedure called "masking" is used in the level set methods to ensure that the value function will not enter in the obstacle region A. Alternatively, numerical tools of [29] for solving variational equations could be used. In both methods, as also stated in [21], at each time-step t and for all grid points x, the value function is computed as $\widetilde{V}(x,t) = \max(h(x), \bar{V}(x,t))$, where $\bar{V}(x,t)$ is the numerical solution of the partial differential equation, which appears as the second term in (11). A similar procedure is followed for $V(x,t)$, where the second term of (4) is solved instead.

C. Simulation Results

For the numerical computation, we used four fixed obstacles and considered $\alpha = 0.2$ m/s, $\beta = 0°$ (aligned with the x-axis) to be the current disturbance amplitude and orientation respectively. The orientation of the underwater vehicle can vary in the interval $[-\pi, \pi]$. For the simulations, $u_{1,\max} = 1.6$ m/s, $u_{1,\min} = -1.6$ m/s, $u_{2,\max} = 0.8$ rad/s, $u_{2,\min} = -0.8$ rad/s, $v_{1,\max} = 0.8$ m/s, and $v_{1,\min} = -0.8$ m/s were chosen from [31] to be the extrema of the control and disturbance inputs. Contour slices of the resulting set $\widetilde{RA}(t,R,A)$, for $t = 3s$ and different values of x_3, are shown in Fig. 1(a)–(c). The reachable sets include all states inside the area determined by the solid lines and, as expected, do not include points inside the fixed obstacles denoted by rectangles. The filled square represents the target set that the vehicle aims to reach, whereas the dashed square indicates the boundary of the motion space. For comparison purposes, the dashed lines depict the reachable sets at $t = 2s$.

So far, the disturbance current was assumed to have constant magnitude and direction. In a worst-case setting, the angle β of the current can be considered as an additional disturbance input $v_2 = \beta$, which is also trying to maximize the Hamiltonian of the system. The maximum value of H is attained for $v_2^* = \arctan 2(p_2, p_1)$. The numerically computed reachable set for this case is depicted in Fig. 2(a). It implies that only the points that belong to this $3D$ set can reach the target for any disturbance direction and for any value of v_1. The transparent cube indicates the boundary of the motion space.

For a more realistic implementation of the worst-case scenario, the state space could be augmented with $\dot{\beta} = v_2 \in [-5°/\text{s}\ 5°/\text{s}]$, and so the derivative of the current's angle, instead of the actual angle, could be treated as an additional disturbance input. Fig. 2(b) depicts a 3-D projection of the 4-D reachable set for $t = 3$ s. This projection represents a union over the reachable sets that correspond to each disturbance angle, and as expected, it is a superset of the one of Fig. 2(a) (conservative case) since the disturbance does not change direction instantaneously any more, and subset of that of Fig. 1(d), where the disturbance was assumed to have constant direction. The main purpose of this example was to illustrate the proposed formulation numerically, and from an application point of view, further investigation is required.

All simulations were performed on an Intel Core 2 Duo 2.66-GHz processor running Windows 7 and using the Level

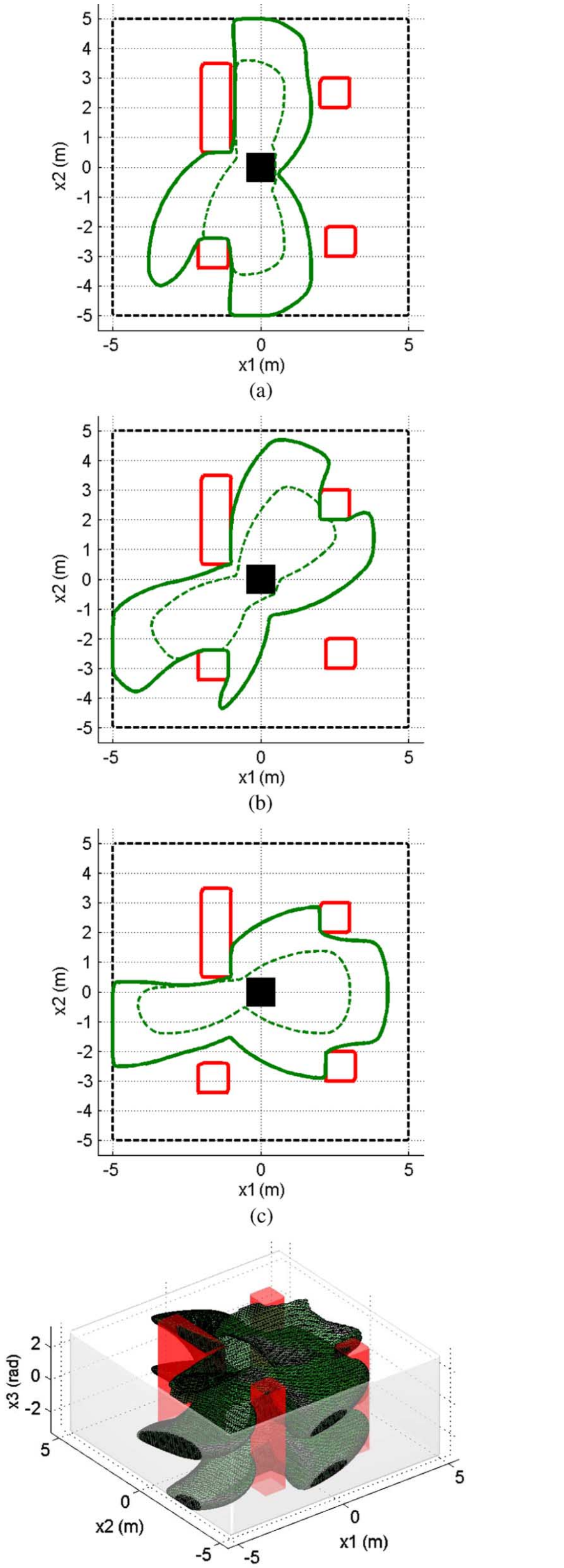

Fig. 1. Contour plots of $\widetilde{RA}(t,R,A)$ for (a) $x_3 = (\pi/2)$ rad, (b) $x_3 = (\pi/4)$ rad, and (c)$x_3 = 0$ rad. (d) 3-D representation of $\widetilde{RA}(t,R,A)$.

Set Method Toolbox [23] (ver. 1.1) on MATLAB 7.10. Since the Level Set Method Toolbox is based on gridding the state space, the memory and computational cost grow exponentially with the dimension of the system, and hence the algorithm suffers from the "curse of dimensionality" [21]. On the other hand, assuming that an accurate enough grid is used, tight approximations of the (in general irregular and nonconvex) reachable sets can be achieved. The details for the numerical implementation of the specific example are summarized in Table I. As expected, the first two cases, which were performed on the same grid, required similar time and memory usage, whereas the 4-D implementation led to a significant increase both in memory and computational time.

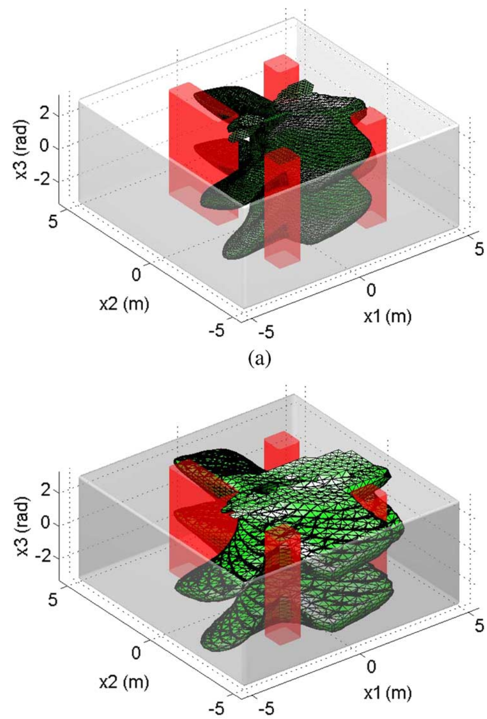

Fig. 2. (a) Worst-case analysis with β as additional disturbance input. (b) Worst-case analysis with $\dot{\beta}$ as additional disturbance input.

TABLE I
NUMERICAL STATISTICS OF THE REACHABILITY COMPUTATIONS

	3D case $\beta = 0$	3D case $v_2 = \beta$	4D case $v_2 = \dot{\beta}$
Time	305.12s	338.45s	970.15s
Memory	216MB	296MB	702MB
Grid	$81 \times 81 \times 81$	$81 \times 81 \times 81$	$31 \times 31 \times 31 \times 81$

V. CONCLUSION

A new framework of controlling nonlinear systems with state constraints and competing inputs was presented, and a proof that the value function of the resulting reach–avoid problem is the viscosity solution to a variational equation was provided. The formulation was based on reachability and game theory and has the advantage of maintaining the continuity in the value function and the Hamiltonian of the system. As a consequence, it has very good numerical properties in the sense that standard numerical tools can be now formally used. The effectiveness of the proposed approach was verified numerically in the target-hitting problem of an underactuated underwater vehicle in the presence of obstacles. For the numerical implementation, standard tools based on Level Set Methods were used.

In future work, we plan to extend the proposed approach to formulate games in the case where the obstacle function is time- and/or control-dependant. Another issue would be to provide a systematic methodology in order to construct numerically the theoretically optimal control policy. This is in general difficult to construct since it requires the computation of derivatives of the value function, which is a process very sensitive to numerical errors. Finally, the developed reach–avoid operator provides an alternative formulation for the viability-based approaches and could be extended and incorporated in existing algorithms for verification of hybrid systems.

Appendix A

A. Proof of Proposition 2

Proof:

Part 1: Following [33, Lemma 8], we first show that $\widetilde{RA}(\tau,R,A) \subseteq \{x \in \mathbb{R}^n | \widetilde{V}(x,\tau) \leq 0\}$. Consider $x \in \widetilde{RA}(\tau,R,A)$, and for the sake of contradiction, assume that $\widetilde{V}(x,t) > 0$. Then, there exists $\epsilon > 0$ such that for all $\tilde{\gamma}(\cdot) \in \widetilde{\Gamma}_{[t,T]}$, $\sup_{v(\cdot)\in\mathcal{V}_{[t,T]}} \max\{l(\tilde{\phi}(T,t,x,\tilde{\gamma}[v](\cdot),v(\cdot)))$, $\max_{\tau\in[t,T]} h(\tilde{\phi}(\tau,t,x,\tilde{\gamma}[v](\cdot),v(\cdot)))\} > 2\epsilon > 0$. This in turn implies that for all $\tilde{\gamma}(\cdot) \in \widetilde{\Gamma}_{[t,T]}$, there exists $\hat{v}(\cdot) \in \mathcal{V}_{[t,T]}$ such that either $l(\tilde{\phi}(T,t,x,\tilde{\gamma}[\hat{v}](\cdot),\hat{v}(\cdot))) > \epsilon > 0$, or there exists $\tau \in [t,T]$ such that $h(\tilde{\phi}(\tau,t,x,\tilde{\gamma}[\hat{v}](\cdot),\hat{v}(\cdot))) > \epsilon > 0$.

Consider now the implications of $x \in \widetilde{RA}(\tau,R,A)$. Equation (5) implies that there exists a $\gamma(\cdot) \in \Gamma_{[t,T]}$ such that for all $v(\cdot) \in \mathcal{V}_{[t,T]}$, and so also for $\hat{v}(\cdot)$, we can define $u(\cdot) = \gamma[\hat{v}](\cdot)$. Then, for this $u(\cdot)$ and $\hat{v}(\cdot)$, there exists $\tau_1 \in [t,T]$ such that $\phi(\tau_1,x,t,u(\cdot),\hat{v}(\cdot)) \in R$ and for all $\tau_2 \in [t,\tau_1]\phi(\tau_2,t,x,u(\cdot),\hat{v}(\cdot)) \notin A$. Choose the freezing input signal as

$$\bar{u}(\tau) = \begin{cases} 1, & \text{for } \tau \in [t,\tau_1] \\ 0, & \text{for } \tau \in [\tau_1,T]. \end{cases}$$

If we combine $\bar{u}(\cdot)$ with $u(\cdot)$, we can get the input $\tilde{u}(\cdot)$, which will generate a trajectory

$$\tilde{\phi}(\tau,x,t,\tilde{u}(\cdot),\hat{v}(\cdot)) = \begin{cases} \phi(\tau,x,t,u(\cdot),\hat{v}(\cdot)), & \text{if } \tau \in [t,\tau_1] \\ \phi(\tau_1,x,t,u(\cdot),\hat{v}(\cdot)), & \text{if } \tau \in [\tau_1,T]. \end{cases} \tag{12}$$

Case 1.1: Consider first the case where for all $\tilde{\gamma}(\cdot) \in \widetilde{\Gamma}_{[t,T]} l(\tilde{\phi}(T,t,x,\tilde{\gamma}[\hat{v}](\cdot),\hat{v}(\cdot))) > \epsilon > 0$. For $\tau = T$, we have that

$$\tilde{\phi}(T,x,t,\tilde{u}(\cdot),\hat{v}(\cdot)) = \phi(\tau_1,x,t,u(\cdot),\hat{v}(\cdot)).$$

Since $x \in \widetilde{RA}(\tau,R,A)$, we showed before that $\phi(\tau_1,x,t,u(\cdot),\hat{v}(\cdot)) \in R$, i.e. $l(\phi(\tau_1,x,t,u(\cdot),\hat{v}(\cdot))) \leq 0$. Thus, from (12) we have that $l(\tilde{\phi}(T,x,t,\tilde{u}(\cdot),\hat{v}(\cdot))) \leq 0$. Since $u(\cdot) = \gamma[\hat{v}](\cdot)$ is already nonanticipative, and a nonanticipative strategy for $\bar{u}(\cdot)$ can be designed, $\tilde{u}(\cdot)$ will also be nonanticipative. Therefore, the previous statement establishes a contradiction.

Case 1.2: Consider now the case where for all $\tilde{\gamma}(\cdot) \in \widetilde{\Gamma}_{[t,T]}$ there exists $\tau \in [t,T]$ such that $h(\tilde{\phi}(\tau,t,x,\tilde{\gamma}[\hat{v}](\cdot),\hat{v}(\cdot))) > \epsilon > 0$. Since we showed that for all $\tau \in [t,\tau_1], \phi(\tau,t,x,u(\cdot),\hat{v}(\cdot)) \notin A$, we can conclude from (12) that for all $\tau \in [t,\tau_1]$

$$h\left(\tilde{\phi}(\tau,x,t,\tilde{u}(\cdot),\hat{v}(\cdot))\right) \leq 0.$$

If $\tau \in [\tau_1,T]$, we have that

$$\tilde{\phi}(\tau,x,t,\tilde{u}(\cdot),\hat{v}(\cdot)) = \phi(\tau_1,x,t,u(\cdot),\hat{v}(\cdot)).$$

Thus, $h(\tilde{\phi}(\tau,x,t,\tilde{u}(\cdot),\hat{v}(\cdot))) = h(\phi(\tau_1,x,t,u(\cdot),\hat{v}(\cdot))) \leq 0$. Hence, for all $\tau \in [t,T]$, we have that $h(\tilde{\phi}(\tau,x,t,\tilde{u}(\cdot),\hat{v}(\cdot))) \leq 0$. Since in Case 1.1 $\tilde{u}(\cdot)$ was shown to be nonanticipative, we have a contradiction.

Part 2: Next, we show that $\{x \in \mathbb{R}^n | \widetilde{V}(x,\tau) \leq 0\} \subseteq \widetilde{RA}(\tau,R,A)$. Consider (x,t) such that $\widetilde{V}(x,t) \leq 0$ and assume for the sake of contradiction that $x \notin \widetilde{RA}(\tau,R,A)$. Then, for all $\gamma(\cdot) \in \Gamma_{[t,T]}$ there exists $\hat{v}(\cdot) \in \mathcal{V}_{[t,T]}$ such that either for all $\tau_1 \in [t,T]$, $\phi(\tau_1,t,x,\gamma(\cdot),v(\cdot)) \notin R$, or there exists $\tau_2 \in [t,T]$ such that $\phi(\tau_2,t,x,\gamma(\cdot),v(\cdot)) \in A \wedge \forall \tau_2' \in [t,\tau_2], \phi(\tau_2',t,x,\gamma(\cdot),v(\cdot)) \notin R$.

Consider the strategy $\tilde{\gamma}(\cdot) \in \widetilde{\Gamma}_{[t,T]}$ (note that $\tilde{\gamma}(\cdot) \in \widetilde{\Gamma}_{[t,T]}$ follows from the implications of $\widetilde{V}(x,t) \leq 0$ and will be defined in the sequel), which consists of a strategy $\gamma(\cdot) \in \Gamma_{[t,T]}$, as defined in Section II-B, and an additional scalar component that corresponds to $\bar{u}$. Following [33, Lemma 8], by eliminating this scalar component, we can extract $\gamma(\cdot) \in \Gamma_{[t,T]}$ from $\tilde{\gamma}(\cdot) \in \widetilde{\Gamma}_{[t,T]}$. By the implications of $x \notin \widetilde{RA}(\tau,R,A)$, we can then choose the $\hat{v}(\cdot) \in \mathcal{V}_{[t,T]}$ that corresponds to that $\gamma(\cdot)$. In [33, Lemma 4], it was proven that the set of states visited by the augmented trajectory is a subset of the states visited by the original one. We therefore have that for all $\tau_1 \in [t,T]$

$$\phi(\tau_1,x,t,\gamma[\hat{v}](\cdot),\hat{v}(\cdot)) \notin R \Longrightarrow \tilde{\phi}(\tau_1,x,t,\tilde{\gamma}[\hat{v}](\cdot),\hat{v}(\cdot)) \notin R \tag{13}$$

or there exists $\tau_2^* \in [t,T]$ such that

$$\phi(\tau_2^*,x,t,\gamma[\hat{v}](\cdot),\hat{v}(\cdot)) \in A \Longrightarrow \tilde{\phi}(\tau_2^*,x,t,\tilde{\gamma}[\hat{v}](\cdot),\hat{v}(\cdot)) \in A \tag{14}$$

and similarly, $\forall \tau_2' \in [t,\tau_2^*], \tilde{\phi}(\tau_2',x,t,\tilde{\gamma}[\hat{v}](\cdot),\hat{v}(\cdot)) \notin R$. By (13) and (14), and based on the definition of R and A, we conclude that there exists a $\delta > 0$ such that either for all $\tau_1 \in [t,T]$

$$l\left(\tilde{\phi}(\tau_1,x,t,\tilde{\gamma}[\hat{v}](\cdot),\hat{v}(\cdot))\right) > \delta > 0 \tag{15}$$

or for some $\tau_2^* \in [t,T]$

$$h\left(\tilde{\phi}(\tau_2^*,x,t,\tilde{\gamma}[\hat{v}](\cdot),\hat{v}(\cdot))\right) > \delta > 0$$
$$\forall \tau_2' \in [t,\tau_2^*]\, l\left(\tilde{\phi}(\tau_2',x,t,\tilde{\gamma}[\hat{v}](\cdot),\hat{v}(\cdot))\right) > \delta > 0. \tag{16}$$

Since $\widetilde{V}(x,t) \leq 0$, then for all $\epsilon > 0$, there exists a nonanticipative strategy $\tilde{\gamma}(\cdot) \in \widetilde{\Gamma}_{[t,T]}$ such that $\sup_{v(\cdot)\in\mathcal{V}_{[t,T]}}, \max\{l(\tilde{\phi}(T,t,x,\tilde{\gamma}[v](\cdot),v(\cdot)))$, $\max_{\tau\in[t,T]} h(\tilde{\phi}(\tau,t,x,\tilde{\gamma}[v](\cdot),v(\cdot)))\} \leq \epsilon$. Hence, for all $v(\cdot) \in \mathcal{V}_{[t,T]}$, $l(\tilde{\phi}(T,t,x,\tilde{\gamma}[v](\cdot),v(\cdot))) \leq \epsilon$, and for all

$\tau \in [t,T]$, $h(\tilde{\phi}(\tau,t,x,\tilde{\gamma}[v](\cdot),v(\cdot))) \leq \epsilon$. For $v(\cdot)=\hat{v}(\cdot)$, the last argument implies that

$$l\left(\tilde{\phi}\left(T,x,t,\tilde{\gamma}[\hat{v}](\cdot),\hat{v}(\cdot)\right)\right) \leq \epsilon$$

and for $\tau=\tau_2^*$

$$h\left(\tilde{\phi}\left(\tau_2^*,x,t,\tilde{\gamma}[\hat{v}](\cdot),\hat{v}(\cdot)\right)\right) \leq \epsilon.$$

If we choose $\epsilon=\delta/2$, the last statements contradict (15) and (16) and complete the proof. ■

Appendix B

A. Proof of Lemma 1

Proof: Following [17, Theorem 3.1], we can define

$$W(x,t)=\inf_{\gamma(\cdot)\in\Gamma_{[t,t+\alpha]}}\sup_{v(\cdot)\in\mathcal{V}_{[t,t+\alpha]}}\left[\max\left\{\max_{\tau\in[t,t+\alpha]}h\left(\phi\left(\tau,t,x,\gamma(\cdot),v(\cdot)\right)\right),V\left(\phi\left(t+\alpha,t,x,\gamma(\cdot),v(\cdot)\right),t+\alpha\right)\right\}\right].$$

We will then show that for all $\epsilon>0$, $V(x,t)\leq W(x,t)+2\epsilon$ and $V(x,t)\geq W(x,t)-3\epsilon$. Then, since $\epsilon>0$ is arbitrary, $V(x,t)=W(x,t)$.

Case 1: $V(x,t)\leq W(x,t)+2\epsilon$. Fix $\epsilon>0$ and choose $\gamma_1(\cdot)\in\Gamma_{[t,t+\alpha]}$ such that

$$W(x,t)\geq\sup_{v_1(\cdot)\in\mathcal{V}_{[t,t+\alpha]}}\left[\max\left\{\max_{\tau\in[t,t+\alpha]}h\left(\phi\left(\tau,t,x,\gamma_1(\cdot),v_1(\cdot)\right)\right),V\left(\phi\left(t+\alpha,t,x,\gamma_1(\cdot),v_1(\cdot)\right),t+\alpha\right)\right\}\right]-\epsilon.$$

Similarly, choose $\gamma_2(\cdot)\in\Gamma_{[t+\alpha,T]}$ such that we have the first equation shown at the bottom of the page. For any $v(\cdot)\in\mathcal{V}_{[t,T]}$, we can define $v_1(\cdot)\in\mathcal{V}_{[t,t+\alpha]}$ and $v_2(\cdot)\in\mathcal{V}_{[t+\alpha,T]}$ such that $v_1(\tau)=v(\tau)$ for all $\tau\in[t,t+\alpha)$ and $v_2(\tau)=v(\tau)$ for all $\tau\in[t+\alpha,T]$. Define also $\gamma(\cdot)\in\Gamma_{[t,T]}$ by

$$\gamma[v](\tau)=\begin{cases}\gamma_1[v_1](\tau), & \text{if } \tau\in[t,t+\alpha)\\ \gamma_2[v_2](\tau), & \text{if } \tau\in[t+\alpha,T].\end{cases}$$

It is easy to see that $\gamma:\mathcal{V}_{[t,T]}\to\mathcal{U}_{[t,T]}$ is nonanticipative. By uniqueness, $\phi(\tau,t,x,\gamma(\cdot),v(\cdot))=\phi(\tau,t,x,\gamma_1(\cdot),v_1(\cdot))$ in case $\tau\in[t,t+\alpha)$, and also $\phi(\tau,t,x,\gamma(\cdot),v(\cdot))=\phi(\tau,t+\alpha,\phi(t+\alpha,t,x,\gamma_1(\cdot),v_1(\cdot)),\gamma_2(\cdot),v_2(\cdot))$ if $\tau\in[t+\alpha,T]$.

Hence, we have the second equation shown at the bottom of the page. Therefore, $V(x,t)\leq W(x,t)+2\epsilon$.

Case 2: $V(x,t)\geq W(x,t)-3\epsilon$. Fix $\epsilon>0$ and choose now $\gamma(\cdot)\in\Gamma_{[t,T]}$ such that

$$V(x,t)\geq\sup_{v(\cdot)\in\mathcal{V}_{[t,T]}}\max\left\{l\left(\phi\left(T,t,x,\gamma(\cdot),v(\cdot)\right)\right),\max_{\tau\in[t,T]}h\left(\phi\left(\tau,t,x,\gamma(\cdot),v(\cdot)\right)\right)\right\}-\epsilon. \quad (17)$$

By the definition of $W(x,t)$

$$W(x,t)\leq\sup_{v(\cdot)\in\mathcal{V}_{[t,t+\alpha]}}\left[\max\left\{\max_{\tau\in[t,t+\alpha]}h(\phi\left(\tau,t,x,\gamma(\cdot),v(\cdot)\right)),V\left(\phi\left(t+\alpha,t,x,\gamma(\cdot),v(\cdot)\right),t+\alpha\right)\right\}\right].$$

$$V\left(\phi\left(t+\alpha,t,x,\gamma_1(\cdot),v_1(\cdot)\right),t+\alpha\right)\geq\sup_{v_2(\cdot)\in\mathcal{V}_{[t+\alpha,T]}}\max\left\{l\left(\phi\left(T,t+\alpha,\phi\left(t+\alpha,t,x,\gamma_1(\cdot),v_1(\cdot)\right),\gamma_2(\cdot),v_2(\cdot)\right)\right),\max_{\tau\in[t+\alpha,T]}h\left(\phi\left(\tau,t+\alpha,\phi\left(t+\alpha,t,x,\gamma_1(\cdot),v_1(\cdot)\right),\gamma_2(\cdot),v_2(\cdot)\right)\right)\right\}-\epsilon.$$

$$\begin{aligned}W(x,t)\geq&\sup_{v_1(\cdot)\in\mathcal{V}_{[t,t+\alpha]}}\sup_{v_2(\cdot)\in\mathcal{V}_{[t+\alpha,T]}}\max\left\{\max_{\tau\in[t,t+\alpha]}h\left(\phi\left(\tau,t,x,\gamma_1(\cdot),v_1(\cdot)\right)\right),\right.\\ &l\left(\phi\left(T,t+\alpha,\phi\left(t+\alpha,t,x,\gamma_1(\cdot),v_1(\cdot)\right),\gamma_2(\cdot),v_2(\cdot)\right)\right),\\ &\left.\max_{\tau\in[t+\alpha,T]}h\left(\phi\left(\tau,t+\alpha,\phi\left(t+\alpha,t,x,\gamma_1(\cdot),v_1(\cdot)\right),\gamma_2(\cdot),v_2(\cdot)\right)\right)\right\}-2\epsilon\\ \geq&\sup_{v(\cdot)\in\mathcal{V}_{[t,T]}}\max\left\{l\left(\phi\left(T,t,x,\gamma(\cdot),v(\cdot)\right)\right),\max_{\tau\in[t,T]}h\left(\phi\left(\tau,t,x,\gamma(\cdot),v(\cdot)\right)\right)\right\}-2\epsilon\\ \geq&V(x,t)-2\epsilon.\end{aligned}$$

Hence, there exists a $v_1(\cdot) \in \mathcal{V}_{[t,t+\alpha]}$ such that

$$W(x,t) \leq \max\left\{ \max_{\tau\in[t,t+\alpha]} h\left(\phi\left(\tau,t,x,\gamma(\cdot),v_1(\cdot)\right)\right), V\left(\phi\left(t+\alpha,t,x,\gamma(\cdot),v_1(\cdot)\right),t+\alpha\right)\right\} + \epsilon. \quad (18)$$

Let $\hat{v}(\tau) = v_1(\tau)$ for all $\tau \in [t,t+\alpha)$, and $\hat{v}(\tau) = v'(\tau)$ for all $\tau \in [t+\alpha,T]$. Let also $\gamma' \in \Gamma_{[t+\alpha,T]}$ to be the restriction of the nonanticipative strategy $\gamma(\cdot)$ over $[t+\alpha,T]$. Then, for all $\tau \in [t+\alpha,T]$, we define $\gamma'[v'](\tau) = \gamma[\hat{v}](\tau)$. Hence, we have

$$\begin{aligned} &V\left(\phi\left(t+\alpha,t,x,\gamma(\cdot),v_1(\cdot)\right),t+\alpha\right) \\ &\leq \sup_{v'(\cdot)\in\mathcal{V}_{[t+\alpha,T]}} \max\left\{ l\left(\phi\left(T,t+\alpha,\phi\left(t+\alpha,t,x,\gamma(\cdot),v_1(\cdot)\right),\gamma'(\cdot),v'(\cdot)\right)\right), \max_{\tau\in[t+\alpha,T]} h\left(\phi\left(\tau,t+\alpha,\phi\left(t+\alpha,t,x,\gamma(\cdot),v_1(\cdot)\right),\gamma'(\cdot),v'(\cdot)\right)\right)\right\}\end{aligned}$$

and so there exists a $v_2(\cdot) \in \mathcal{V}_{[t+\alpha,T]}$ such that

$$\begin{aligned} &V\left(\phi\left(t+\alpha,t,x,\gamma(\cdot),v_1(\cdot)\right),t+\alpha\right) \\ &\quad\leq \max\left\{ l\left(\phi\left(T,t+\alpha,\phi\left(t+\alpha,t,x,\gamma(\cdot),v_1(\cdot)\right),\gamma'(\cdot),v_2(\cdot)\right)\right),\right. \\ &\quad\quad \left.\max_{\tau\in[t+\alpha,T]} h(\phi\left(\tau,t+\alpha,\phi(t+\alpha,t,x,\gamma(\cdot),v_1(\cdot)),\gamma'(\cdot),v_2(\cdot)\right))\right\} + \epsilon. \quad (19)\end{aligned}$$

We can define

$$v(\tau) = \begin{cases} v_1(\tau), & \text{if } \tau \in [t,t+\alpha) \\ v_2(\tau), & \text{if } \tau \in [t+\alpha,T]. \end{cases}$$

Therefore, from (18) and (19)

$$W(x,t) \leq \max\left\{ l\left(\phi\left(T,t,x,\gamma(\cdot),v(\cdot)\right)\right), \max_{\tau\in[t,T]} h\left(\phi\left(\tau,t,x,\gamma(\cdot),v(\cdot)\right)\right)\right\} + 2\epsilon$$

which together with (17) implies $V(x,t) \geq W(x,t) - 3\epsilon$. ■

B. Proof of Lemma 2

Proof: Since l and h are bounded, V is also bounded. For the second part fix $x, \hat{x} \in \mathbb{R}^n$ and $t \in [0,T]$. Let $\epsilon > 0$ and choose $\hat{\gamma}(\cdot) \in \Gamma_{[t,T]}$ such that

$$V(\hat{x},t) \geq \sup_{v(\cdot)\in\mathcal{V}_{[t,T]}} \max_{\tau\in[t,T]} \max\left\{ l\left(\phi\left(T,t,\hat{x},\hat{\gamma}(\cdot),v(\cdot)\right)\right), h\left(\phi\left(\tau,t,\hat{x},\hat{\gamma}(\cdot),v(\cdot)\right)\right)\right\} - \epsilon.$$

By definition

$$V(x,t) \leq \sup_{v(\cdot)\in\mathcal{V}_{[t,T]}} \max_{\tau\in[t,T]} \max\left\{ l\left(\phi\left(T,t,x,\hat{\gamma}(\cdot),v(\cdot)\right)\right), h\left(\phi\left(\tau,t,x,\hat{\gamma}(\cdot),v(\cdot)\right)\right)\right\}.$$

We can choose $\hat{v}(\cdot) \in \mathcal{V}_{[t,T]}$ such that

$$V(x,t) \leq \max_{\tau\in[t,T]} \max\left\{ l\left(\phi\left(T,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right), h\left(\phi\left(\tau,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right)\right\} + \epsilon$$

and hence

$$\begin{aligned} &V(x,t) - V(\hat{x},t) \\ &\quad\leq \max_{\tau\in[t,T]} \max\left\{ l\left(\phi\left(T,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right), h\left(\phi\left(\tau,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right)\right\} \\ &\quad\quad - \max_{\tau\in[t,T]} \max\left\{ l\left(\phi\left(T,t,\hat{x},\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right), h\left(\phi\left(\tau,t,\hat{x},\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right)\right\} + 2\epsilon.\end{aligned}$$

For all $\tau \in [t,T]$

$$\begin{aligned} &\left|\phi\left(\tau,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right) - \phi\left(T,t,\hat{x},\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right| \\ &\quad= \left|(x-\hat{x}) + \int_t^T \left[f\left(\phi\left(s,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right) - f\left(\phi\left(s,t,\hat{x},\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right)\right] ds\right| \\ &\quad\leq |x-\hat{x}| + \int_t^T \left|f\left(\phi\left(s,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right) - f\left(\phi\left(s,t,\hat{x},\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right)\right| ds \\ &\quad\leq |x-\hat{x}| + C_f \int_t^T \left|\phi\left(s,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right) - \phi\left(s,t,\hat{x},\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right| ds\end{aligned}$$

where C_f is the Lipschitz constant of f. By the Gronwall–Bellman Lemma [36, p. 86], there exists a constant $C_x > 0$ such that for all $\tau \in [t,T]$

$$\left|\phi\left(\tau,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right) - \phi\left(T,t,\hat{x},\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right| \leq C_x|x-\hat{x}|.$$

Let $\tau_0 \in [t,T]$ be such that

$$h\left(\phi\left(\tau_0,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right) = \max_{\tau\in[t,T]} h\left(\phi\left(\tau,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right).$$

Then

$$\begin{aligned} &V(x,t) - V(\hat{x},t) \\ &\quad\leq \max\left\{ l\left(\phi\left(T,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right), h\left(\phi\left(\tau_0,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right)\right\} \\ &\quad\quad - \max\left\{ l\left(\phi\left(T,t,\hat{x},\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right), h\left(\phi\left(\tau_0,t,\hat{x},\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right)\right\} + 2\epsilon.\end{aligned}$$

Case 1: $l(\phi(T,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot))) \geq h(\phi(\tau_0,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)))$

$$\begin{aligned}V(x,t) - V(\hat{x},t) &\leq l\left(\phi\left(T,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right) \\ &\quad - \max\left\{l\left(\phi\left(T,t,\hat{x},\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right), h\left(\phi\left(\tau_0,t,\hat{x},\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right)\right\} + 2\epsilon \\ &\leq l\left(\phi\left(T,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right) - l\left(\phi\left(T,t,\hat{x},\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right) + 2\epsilon \\ &\leq C_l C_x |x-\hat{x}| + 2\epsilon.\end{aligned}$$

Case 2: $l(\phi(T,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot))) < h(\phi(\tau_0,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)))$

$$\begin{aligned}V(x,t) - V(\hat{x},t) &\leq h\left(\phi\left(\tau_0,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right) \\ &\quad - \max\left\{l\left(\phi\left(T,t,\hat{x},\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right), h\left(\phi\left(\tau_0,t,\hat{x},\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right)\right\} + 2\epsilon \\ &\leq h\left(\phi\left(\tau_0,t,x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right) - h\left(\phi\left(\tau_0,t,\hat{x},\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right) + 2\epsilon \\ &\leq C_h C_x |x-\hat{x}| + 2\epsilon.\end{aligned}$$

Thus, in any case $V(x,t) - V(\hat{x},t) \leq \max\{C_l C_h\} C_x |x-\hat{x}| + 2\epsilon$. The same argument with the roles of x, $\hat{x}$ reversed establishes that $V(\hat{x},t) - V(x,t) \leq \max\{C_l C_h\} C_x |x-\hat{x}| + 2\epsilon$. Since ϵ is arbitrary

$$|V(x,t) - V(\hat{x},t)| \leq \max\{C_l C_h\} C_x |x-\hat{x}|.$$

Finally, consider $x \in \mathbb{R}^n$ and $t,\hat{t} \in [0,T]$. Without loss of generality, assume that $t < \hat{t}$. Let $\epsilon > 0$ and choose $\gamma(\cdot) \in \Gamma_{[t,T]}$ such that

$$\begin{aligned}V(x,t) &\geq \sup_{v(\cdot)\in\mathcal{V}_{[t,T]}} \max_{\tau\in[t,T]} \max\left\{l\left(\phi\left(T,t,x,\gamma(\cdot),v(\cdot)\right)\right), h\left(\phi\left(\tau,t,x,\gamma(\cdot),v(\cdot)\right)\right)\right\} - \epsilon \\ &\geq \max_{\tau\in[t,T]} \max\left\{l\left(\phi\left(T,t,x,\gamma(\cdot),v(\cdot)\right)\right), h\left(\phi\left(\tau,t,x,\gamma(\cdot),v(\cdot)\right)\right)\right\} - \epsilon.\end{aligned}$$

By definition

$$V(x,\hat{t}) \leq \sup_{v(\cdot)\in\mathcal{V}_{[\hat{t},T]}} \max_{\tau\in[\hat{t},T]} \max\left\{l\left(\phi\left(T,\hat{t},x,\hat{\gamma}(\cdot),v(\cdot)\right)\right), h\left(\phi\left(\tau,\hat{t},x,\hat{\gamma}(\cdot),v(\cdot)\right)\right)\right\}.$$

Therefore, we can choose $\hat{v}(\cdot) \in \mathcal{V}_{[\hat{t},T]}$ such that

$$V(x,\hat{t}) \leq \max_{\tau\in[\hat{t},T]} \max\left\{l\left(\phi\left(T,\hat{t},x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right), h\left(\phi\left(\tau,\hat{t},x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right)\right\} + \epsilon$$

where $\hat{\gamma} \in \Gamma_{[\hat{t},T]}$ is the restriction of $\gamma(\cdot)$ over $[\hat{t},T]$. Then, for all $\tau \in [\hat{t},T]$, we define $\hat{\gamma}[\hat{v}](\tau) = \gamma[v](\tau)$, and $\hat{v}(\tau) = v(\tau + t - \hat{t})$. By uniqueness, for all $\tau \in [\hat{t},T]$ we have that $\phi(\tau,\hat{t},x,\hat{\gamma}(\cdot),\hat{v}(\cdot)) = \phi(\tau + t - \hat{t},t,x,\gamma(\cdot),v(\cdot))$

$$\begin{aligned}V(x,t) - V(x,\hat{t}) &\geq \max_{\tau\in[t,T]} \max\left\{l\left(\phi\left(T,t,x,\gamma(\cdot),v(\cdot)\right)\right), h\left(\phi\left(\tau,t,x,\gamma(\cdot),v(\cdot)\right)\right)\right\} \\ &\quad - \max_{\tau\in[\hat{t},T]} \max\left\{l\left(\phi\left(T,\hat{t},x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right), h\left(\phi\left(\tau,\hat{t},x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right)\right\} - 2\epsilon.\end{aligned}$$

Case 1: Assume that $l(\phi(T,\hat{t},x,\hat{\gamma}(\cdot),\hat{v}(\cdot))) \geq \max_{\tau\in[\hat{t},T]} h(\phi(\tau,\hat{t},x,\hat{\gamma}(\cdot),\hat{v}(\cdot)))$

$$\begin{aligned}V(x,t) - V(x,\hat{t}) &\geq \max_{\tau\in[t,T]} \max\left\{l\left(\phi\left(T,t,x,\gamma(\cdot),v(\cdot)\right)\right), h\left(\phi\left(\tau,t,x,\gamma(\cdot),v(\cdot)\right)\right)\right\} \\ &\quad - l\left(\phi\left(T,\hat{t},x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right) - 2\epsilon \\ &\geq l\left(\phi\left(T,t,x,\gamma(\cdot),v(\cdot)\right)\right) - l\left(\phi\left(T,\hat{t},x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right) - 2\epsilon \\ &= l\left(\phi\left(T,t,x,\gamma(\cdot),v(\cdot)\right)\right) \\ &\quad - l\left(\phi\left(T+t-\hat{t},t,x,\gamma(\cdot),v(\cdot)\right)\right) - 2\epsilon \\ &\geq -C_l C_f |T - T - t + \hat{t}| - 2\epsilon \\ &= -C_l C_f |\hat{t} - t| - 2\epsilon\end{aligned}$$

where C_l is the Lipschitz constant of l.

Case 2: Assume that $l(\phi(T,\hat{t},x,\hat{\gamma}(\cdot),\hat{v}(\cdot))) < \max_{\tau\in[\hat{t},T]} h(\phi(\tau,\hat{t},x,\hat{\gamma}(\cdot),\hat{v}(\cdot)))$

$$\begin{aligned}V(x,t) - V(x,\hat{t}) &\geq \max_{\tau\in[t,T]} \max\left\{l\left(\phi\left(T,t,x,\gamma(\cdot),v(\cdot)\right)\right), h\left(\phi\left(\tau,t,x,\gamma(\cdot),v(\cdot)\right)\right)\right\} \\ &\quad - \max_{\tau\in[\hat{t},T]} h\left(\phi\left(\tau,\hat{t},x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right) - 2\epsilon \\ &\geq \max_{\tau\in[t,T]} h\left(\phi\left(\tau,t,x,\gamma(\cdot),v(\cdot)\right)\right) \\ &\quad - \max_{\tau\in[\hat{t},T]} h\left(\phi\left(\tau,\hat{t},x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right) - 2\epsilon.\end{aligned}$$

Let $\tau_0 \in [\hat{t},T]$ be such that

$$h\left(\phi\left(\tau_0,\hat{t},x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right) = \max_{\tau\in[\hat{t},T]} h\left(\phi\left(\tau,\hat{t},x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right).$$

Then

$$\begin{aligned}V(x,t) - V(x,\hat{t}) &\geq \max_{\tau\in[t,T]} h\left(\phi\left(\tau,t,x,\gamma(\cdot),v(\cdot)\right)\right) \\ &\quad - h\left(\phi\left(\tau_0,\hat{t},x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right) - 2\epsilon \\ &\geq h\left(\phi\left(\tau_0,t,x,\gamma(\cdot),v(\cdot)\right)\right) \\ &\quad - h\left(\phi\left(\tau_0,\hat{t},x,\hat{\gamma}(\cdot),\hat{v}(\cdot)\right)\right) - 2\epsilon \\ &= h\left(\phi\left(\tau_0,t,x,\gamma(\cdot),v(\cdot)\right)\right) \\ &\quad - h\left(\phi\left(\tau_0+t-\hat{t},t,x,\gamma(\cdot),v(\cdot)\right)\right) - 2\epsilon \\ &\geq -C_h C_f |\tau_0 - \tau_0 - t + \hat{t}| - 2\epsilon \\ &= -C_h C_f |\hat{t} - t| - 2\epsilon\end{aligned}$$

where C_h is the Lipschitz constant of h. In any case we have that

$$V(x,t) - V(x,\hat{t}) \geq -\max\{C_l, C_h\} C_f |\hat{t} - t| - 2\epsilon.$$

A symmetric argument shows that $V(x,t) - V(x,\hat{t}) \leq \max\{C_l, C_h\} C_f |\hat{t} - t| + 2\epsilon$, and since ϵ is arbitrary, this concludes the proof. ■

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