Model Predictive Control with Multiple Constraint Horizons

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***Abstract*— In this work we propose a Model Predictive Control (MPC) formulation that splits constraints in two different types. Motivated by safety considerations, the first type of constraint enforces a control-invariant set, while the second type could represent a less restrictive constraint on the system state. This distinction enables closed-loop sub-optimality results for nonlinear MPC with heterogeneous state constraints (distinct constraints across open loop predicted states), and no terminal elements. Removing the non-invariant constraint recovers the partially constrained case. Beyond its theoretical interest, heterogeneous constrained MPC shows how constraint choices shape the system's closed loop. In the partially constrained case, adjusting the constraint horizon (how many predicted-state constraints are enforced) trades estimation accuracy for computational cost. Our analysis yields first, a sub-optimality upper-bound accounting for distinct constraint sets, their horizons and decay rates, that is tighter for short horizons than prior work. Second, to our knowledge, we give the first lower bound (beyond open-loop cost) on closed-loop sub-optimality. Together these bounds provide a powerful analysis framework, allowing designers to evaluate the effect of horizons in MPC sub-optimality. We demonstrate our results via simulations on nonlinear and linear safety-critical systems.**

***Index Terms*— Model Predictive Control, Optimal Control, Constrained Control, Nonlinear Control.**

I. INTRODUCTION

MODEL Predictive Control (MPC) [1], [2] is an optimization-based method that approximates the infinite-horizon constrained problem by a finite horizon one [3], raising the question of how similar these two solution are. This assessment is done via the MPC's closed-loop suboptimality analysis [4].

An early contribution on closed-loop optimality analysis is [5], which derived bounds on closed-loop suboptimality for linear discrete-time systems by assuming finite-horizon optimal value functions availability. Earlier results are extended in [6] to nonlinear systems, where open-loop value functions are used to evaluate closed-loop suboptimality, yielding a sufficient stability condition explicit in the prediction horizon. Previous works focused on the prediction horizon as the tuning dial for sub-optimality estimation. Sub-optimality was analyzed in [7], for open-loop costs upper-bounded by a chosen running cost within the horizon, assuming uniform constraints and access to optimal state–input pairs. Flexibility on when to start bounding the open-loop cost adds another angle to the suboptimality estimation. Subsequent literature, including [4], [8], [9], move from uniform to dynamic ratios between the open-loop value function and running cost, enabling tighter suboptimality bounds by capturing changes in open and closed-loop costs with greater precision. Extending the knowledge of MPC parameters' effect on sub-optimality, [8] studies the effect of using a control horizon larger than one in closed loop, relevant to networked MPC, with communication delays and data losses [10]–[12]. Sub-optimality was also studied for economic MPC (e.g., [13], [14]).

Recent developments in safety-critical control have sparked a growing interest in enforcing set invariance [15], particularly through the Control Barrier Function (CBF) framework [16]–[18]. This form of constraint enforcement is especially well-suited to receding horizon techniques, which are themselves advantageous in safety-critical problems due to their predictive capabilities [19], [20]. Consequently, MPC-like controllers that integrate CBF constraints have gained traction, with efforts spanning centralized [21], [22], distributed [23], probabilistic [24], learning [25], and real-time [26] MPC frameworks to name a few.

In the wake of this trend and considering CBF as an invariant constraint along the MPC horizon, our earlier work [27] introduced, to the best of our knowledge, the first sub-optimality estimation for "partially constrained" MPC. Relaxing the need to enforce constraints across the full prediction horizon yields results similar to [6], [7], it produced a comparable explicit sub-optimality bound. However, our focus on the role of the constraint horizon and its effect on sub-optimality and stability offers a new perspective on the topic. In this work we enhance the sub-optimality estimation landscape by providing a generalized MPC formulation for discrete-time nonlinear systems where multiple state constraints are applied over the horizon. Our contributions are threefold. First, our formulation contributes conceptually to MPC frameworks without terminal penalty and constraints such as [4], [28]–[30], analyzing the case of different types of constraint hori-

Submission for review: dd/mm/yyyy. AAdN, AP and KM acknowledge funding support by MathWorks. AP was supported in part by UK's Engineering and Physical Sciences Research Council projects EP/X017982/1 and EP/Y014073/1. For the purpose of Open Access, the authors have applied a CC BY public copyright licence to any Author Accepted Manuscript (AAM) version arising from this submission.

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zons. If the second horizon in the problem is considered to be unconstrained, the formulation reduces to a partially constrained one. Second, differently from [6]–[9], [31], we derive closed-loop cost upper-bound depending explicitly on the prediction and constraint horizons. To do so, a difference of value functions distinguishing between different constraint sets is proposed, yielding both implicit and explicit bounds by leveraging constraint and convergence rates. If compared to [27], which adopts a similar setting, our method leads to tighter bounds with a broader validity region. This is also a tool intuitively connecting the effect of extra constraints on performance and stability. Third, to our knowledge, compared to cited works, we provide the first lower-bound on the closed-loop cost that can be tighter (condition dependent) than the finite horizon open-loop cost. This novel lower-bound computation method also issues a "certificate" when it cannot find a tighter bound, validating the "naive open loop cost" as reasonable. For lower bounds emulating an infinite open loop cost via long prediction horizons, we offer a computationally cheaper alternative.

This paper is structured as follows: Section II presents the problem formulation, Section III presents and derives related results on Relaxed Dynamic Programming (RDP) which are the building blocks for the next sections. Sections IV and V present results on closed-loop optimality upper and lower bounds respectively. Section VI displays two distinct numerical simulation cases. The first applies results for a six-degrees-of-freedom nonlinear system, while the second applies it to a linear safety-critical setting. Section VII concludes this work.

II. Problem Formulation

Consider compact sets with non-empty interior $\mathcal{D} \subset \mathbb{R}^n$, $\mathcal{U} \subset \mathbb{R}^m$ and the discrete-time dynamical system

$$x_{k+1} = f(x_k, u_k), \tag{1}$$

where $x_k \in \mathcal{D}$, $u_k \in \mathcal{U}$ and $f(0,0) = 0$. We consider an MPC problem at time k with state x_k, where state constraints over the horizon N are split in two: the first $N - \tilde{N}$ predicted states are constrained by the *control-invariant set* $\mathcal{X}_1$, and the last $\tilde{N}$ predicted states are constrained by a (not necessarily invariant) set $\mathcal{X}_2$, with $\mathcal{X}_1 \subset \mathcal{X}_2 \subseteq \mathcal{D}$.

Remark 1: The relation $\mathcal{X}_1 \subset \mathcal{X}_2$ follows from assuming $\mathcal{X}_1$ is invariant while $\mathcal{X}_2$ need not be. $\mathcal{X}_2 = \mathcal{D}$ covers the "unconstrained" (from the state perspective) tail of the horizon, encompassing [27]. Under these conditions, allowing the case where $\mathcal{X}_2 \subseteq \mathcal{X}_1$, would imply invariance of $\mathcal{X}_2$, which contradicts our assumptions.

The heterogeneously constrained MPC (HC-MPC) problem is formulated as

$$V_N^{\tilde{N}}(x_k) := \min_{u_h(n|x_k)} \sum_{n=0}^{N-1} l(x_h(n|x_k), u_h(n|x_k)) \tag{2a}$$

$$\text{subject to :}$$

$$x_h(n+1|x_k) = f(x_h(n|x_k), u_h(n|x_k)), n = 0, \dots, N-1, \tag{2b}$$

$$x_h(0|x_k) = x_k, \tag{2c}$$

$$u_h(n|x_k) \in \mathcal{U}, n = 0, \dots, N-1, \tag{2d}$$

$$x_h(n+1|x_k) \in \mathcal{X}_1, n = 0, \dots, N - \tilde{N}, \tag{2e}$$

$$x_h(n+1|x_k) \in \mathcal{X}_2, n = N - \tilde{N} + 1, \dots, N-1. \tag{2f}$$

We introduce an auxiliary formulation, the uniformly constrained MPC (UC-MPC), which will be used for analysis purposes and bound derivations at later Sections. UC-MPC is *not* used to compute the control input, whereas the online control is *always* obtained by solving Problem (2).

$$V_N(x_k) := \min_{u_d(n|x_k)} \sum_{n=0}^{N-1} l(x_d(n|x_k), u_d(n|x_k)) \tag{3a}$$

$$\text{subject to :}$$

$$x_d(n+1|x_k) = f(x_d(n|x_k), u_d(n|x_k)), n = 0, \dots, N-1, \tag{3b}$$

$$x_d(0|x_k) = x_k, \tag{3c}$$

$$u_d(n|x_k) \in \mathcal{U}, n = 0, \dots, N-1, \tag{3d}$$

$$x_d(n+1|x_k) \in \mathcal{X}_2, n = 0, \dots, N-1. \tag{3e}$$

We will consider Problems (2) and (3) with $N \geq 2, 2 \leq \tilde{N} \leq N$ and $N \geq 1$ respectively. Note that (3) cannot be derived from (2), as it requires $\tilde{N} = N + 1$ in (2), which is invalid by assumption for (2), as $\tilde{N} \leq N$. Thus, $\tilde{N}$ does not appear in the notation of (3). We assume $x_k \in \mathcal{X}_1$, but we do not require any predicted state of (3) to belong to $\mathcal{X}_1$. Unless stated otherwise, we adopt the following assumptions in this work:

Assumption 1 (Positive definiteness of cost): The cost $l(x, u)$ is assumed to be continuous and positive definite for all $x \in \mathcal{D}$ and for all $u \in \mathcal{U}$, jointly with respect to both arguments, meaning that $l(\cdot, \cdot) > 0$ for all $x, u \neq 0$ and $l(\cdot, \cdot) = 0$ for $x = 0$, and $u = 0$.

Assumption 2 (Viability [4]): For any $x_k \in \mathcal{X}_1, k = \mathbb{N} \cup \{0\}$, we assume problems (2) with $N \geq 2, 2 \leq \tilde{N} \leq N$ and (3) with $N \geq 1$ are feasible and their minima can be attained.

Viability is standard for MPC sub-optimality analysis [4], [6], [32]. Assumption 2 is slightly stronger as we require $\mathcal{X}_1$ to be invariant. Assumption 2 implies that feasibility can be guaranteed along the closed-loop evolution of the system. When it comes to $\mathcal{X}_2$, viability is weaker than requiring its invariance[1] [4]. We highlight that x_k is the measured state describing the actual System (1), while $x_h(n|x_k)$ in (2) and $x_d(n|x_k)$ in (3) are open loop predicted states n steps ahead of the measured state x_k. Let $[u_h^*(0|x_k), \dots, u_h^*(N-1|x_k)]$ be the *open-loop optimal control sequence*, generating the *open-loop optimal trajectory* $[x_h^*(1|x_k), \dots, x_h^*(N|x_k)]$ for problem (2). For future analysis, we introduce their respective counterparts $[u_d^*(0|x_k), \dots, u_d^*(N-1|x_k)]$ and $[x_d^*(1|x_k), \dots, x_d^*(N|x_k)]$ obtained from (3). Let

$$\mu_N^{\tilde{N}}(x_k) := u_h^*(0|x_k), \tag{4}$$

be the *closed-loop* controller, defined to be $\mu_N^{\tilde{N}}(x_k)$ and *always* obtained by the solution of (2). This controller is applied to system (1), producing the closed-loop system dynamics

$$x_{k+1} = f(x_k, \mu_N^{\tilde{N}}(x_k)). \tag{5}$$

[1]Instead of invariance, we require the predicted open-loop trajectory to remain within $\mathcal{X}_2$.

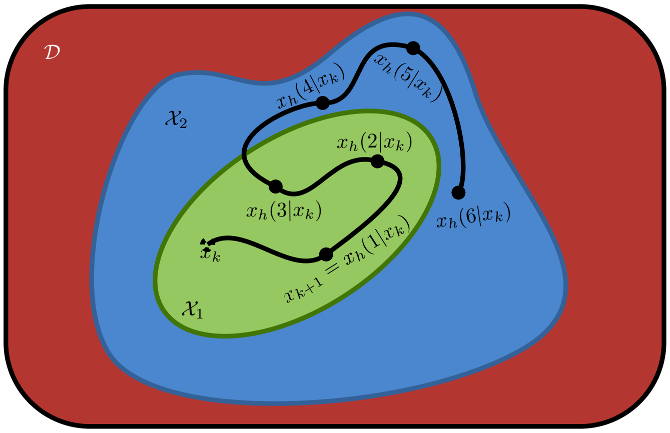


Fig. 1. System (1) under MPC (2) for $\boldsymbol{N = 6}$, $\boldsymbol{N - \tilde{N} = 2}$. Input $\boldsymbol{u_h(2|x_k)}$ produces the last open loop state $\boldsymbol{x_h(3|x_k)}$ in $\boldsymbol{\mathcal{X}_1}$. Subsequent open loop states are subject to $\boldsymbol{\mathcal{X}_2}$. Closed loop states are depicted by $\boldsymbol{x_k}$ and $\boldsymbol{x_{k+1}}$.

At the next time iteration $k+1$, the optimal control problem (2) is solved again, this time for an initial state x_{k+1}. This process is repeated for each new time step and state available. Figure 1 depicts the system behaviour under controller (2).

The associated *closed-loop infinite horizon cost* of (2) is defined by

$$J_{\infty}^{N,\tilde{N}}(x_0) = \sum_{k=0}^{\infty} l(x_k, \mu_N^{\tilde{N}}(x_k)). \tag{6}$$

Our goal is to investigate upper and lower bounds for $J_{\infty}^{N,\tilde{N}}(x_0)$ and explore the impact of the heterogeneous constraint scheme presented. Beyond the tighter upper-bound results and theoretical novelty of the lower-bound per-se, obtaining both bounds works as a powerful design tool. Consider for instance two prediction and constraint horizon pairs $(N, \tilde{N}_1)$ and $(N, \tilde{N}_2)$. We are also interested in providing an answer on which pair performs best in closed loop. We start our investigation by deriving bounds using RDP.

III. Closed Loop Bounds Based on Relaxed Dynamic Programming

A. Closed loop upper-bound

We start by revisiting the following result.

Lemma 1: [6, Proposition 2.2] Consider $N \geq 2, 2 \leq \tilde{N} \leq N$. Let the following Relaxed Dynamic Programming (RDP) equation

$$V_N^{\tilde{N}}(x_k) \geq V_N^{\tilde{N}}(x_{k+1}) + \alpha l(x_k, \mu_N^{\tilde{N}}(x_k)), \tag{7}$$

hold for some $\alpha \in [0, 1]$ and for all $x_k \in \mathcal{X}_1$. Then

$$\alpha J_{\infty}^{N,\tilde{N}}(x_k) \leq V_N^{\tilde{N}}(x_k), \tag{8}$$

holds for all $x_k \in \mathcal{X}_1$.

We now prove that (8), is an upper bound on (6).

Proof: Consider $V_N^{\tilde{N}}(x_k)$ satisfying (7). Equation (7) produces (8) by rewriting it (see [6]):

$$V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(x_{k+1}) \geq \alpha l(x_k, \mu_N^{\tilde{N}}(x_k)). \tag{9}$$

Summing (9) over M sequential discrete time instances:

$$\begin{aligned} V_N^{\tilde{N}}(x_k) &\geq V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(x_{k+1+M}) \\ &\geq \alpha \sum_{j=0}^{M} l(x_{k+j}, \mu_N^{\tilde{N}}(x_{k+j})) = \alpha J_M^{N,\tilde{N}}(x_k). \end{aligned} \tag{10}$$

By letting $M \to \infty$ we obtain (8). ■

The above result holds for general costs. Nonetheless, for positive definite costs on both variables we can obtain the same result without the need to upper bound $V_N^{\tilde{N}}(x_k) \geq V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(x_{k+1+M})$ in (10). This is demonstrated in Lemma 2 (proof in Section A of the Appendix).

Lemma 2: Consider $N \geq 2, 2 \leq \tilde{N} \leq N$, and for all $x_k \in \mathcal{X}_1$, (7) holds for some $\alpha \in [0, 1]$. Furthermore, consider Assumption 1. Then, if $M \to \infty$, $V_N^{\tilde{N}}(x_{k+1+M}) \to 0$.

The restriction to stabilizing MPC problems (the cost $l(\cdot, \cdot)$ is positive definite) is essential here. Under these circumstances, we can also justify Lemma 2 since $V_N^{\tilde{N}}(x_k)$ is a Lyapunov function, and given the negative definiteness of $-\alpha l(x_k, \mu_N^{\tilde{N}}(x_k)) \geq V_N^{\tilde{N}}(x_{k+1}) - V_N^{\tilde{N}}(x_k)$, we can state that when $M \to \infty$, $V_N^{\tilde{N}}(x_{k+1+M}) \to 0$.

B. Closed loop lower-bound

When it comes to lower bounding (6), a naive lower-bound choice can be $V_{\infty}^2(x_k)$, analogous to the lower bound for the traditional problem formulation (3) $V_{\infty}(x_k)$ (as seen in e.g. [6, Prop. 2.2], [7, Lemma 4]). However, since $V_{\infty}^2(x_k)$ can only be approximated by solving (2) with a very large horizon $N \gg \tilde{N}$ (as it is for (3)), its computation is often expensive. A looser but computationally cheaper lower bound is, $V_N^{\tilde{N}}(x_k)$, which holds via

$$J_{\infty}^{N,\tilde{N}}(x_k) \geq V_{\infty}^2(x_k) \geq V_{\infty}^{\tilde{N}}(x_k) \geq V_N^{\tilde{N}}(x_k), \tag{11}$$

and is valid as our problem formulations do not use a terminal cost [6]. We seek a higher lower bound on (6) than $V_N^{\tilde{N}}(x_k)$. Via RDP, we derive results analogous to the upper bound (8).

Proposition 1: Consider $N \geq 2, 2 \leq \tilde{N} \leq N$. Let the following relaxed dynamic programming hold for $\alpha \in [0, 1], \omega \in [0, 1)$, and for all x_k in $\mathcal{X}_1$

$$\alpha l(x_k, \mu_N^{\tilde{N}}(x_k)) \geq \frac{\alpha}{1-\omega}\left[V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(f(x_k, \mu_N^{\tilde{N}}(x_k)))\right]. \tag{12a}$$

Then:

$$\alpha J_{\infty}^{N,\tilde{N}}(x_k) \geq \alpha \frac{V_N^{\tilde{N}}(x_k)}{1-\omega}. \tag{12b}$$

Furthermore, if in addition (7) is fulfilled for all x_k in $\mathcal{X}_1$, then

$$V_N^{\tilde{N}}(x_k) \geq \alpha J_{\infty}^{N,\tilde{N}}(x_k) \geq \frac{\alpha}{1-\omega} V_N^{\tilde{N}}(x_k), \tag{12c}$$

holds for all $x_k \in \mathcal{X}_1$.

The proof of Proposition 1 is in Appendix A. Based on Equation (12c) a necessary condition on ω, is $1 - \omega \geq \alpha$.

Sections IV and V characterize α and ω, respectively: First via direct solution of (2) and (3) and, under extra cost assumptions, explicitly in terms of cost properties and the prediction and constraint horizons.

IV. Closed loop performance upper bound

A. Calculation of α as a difference of value functions

Our goal here is to express α as a function of $(N, \tilde{N})$, which will happen at first implicitly, based on a difference of value functions. This requires analysis of $V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(x_{k+1})$. We start by defining by the optimal open-loop state–input pair of the value functions $V_N^{\tilde{N}}(x_k)$ and $V_N(x_k)$:

$$V_N^{\tilde{N}}(x_k) = \sum_{n=0}^{N-\tilde{N}} \lambda_{h_1}(n|x_k) + \sum_{n=N-\tilde{N}+1}^{N-1} \lambda_{h_2}(n|x_k), \quad (13a)$$

$$\lambda_{h_1}(n|x_k) = l(x_h^*(n|x_k), u_h^*(n|x_k)), n = 0, \dots, N-\tilde{N}, \quad (13b)$$

$$\lambda_{h_2}(n|x_k) = l(x_h^*(n|x_k), u_h^*(n|x_k)), \quad (13c)$$
$$n = N-\tilde{N}+1, \dots, N-1.$$

$$V_N(x_k) = \sum_{n=0}^{N-1} \lambda_d(n|x_k), \quad (14a)$$

$$\lambda_d(n|x_k) = l(x_d^*(n|x_k), u_d^*(n|x_k)), n = 0, \dots, N-1. \quad (14b)$$

The distinction between $\lambda_{h_1}(n|x_k)$ and $\lambda_{h_2}(n|x_k)$ is that $\lambda_{h_1}(n|x_k) = l(x_h^*(n|x_k), u_h^*(n|x_k))$ will use an input $u_h^*(n|x_k)$ which is able to produce $x_h^*(n+1|x_k) \in \mathcal{X}_1$, whereas $\lambda_{h_2}(n|x_k) = l(x_h^*(n|x_k), u_h^*(n|x_k))$ uses a $u_h^*(n|x_k)$ enforcing $x_h^*(n+1|x_k) \in \mathcal{X}_2$. As $\lambda_d(n|x_k)$ uses $u_d^*(n|x_k)$ always generating states in $\mathcal{X}_2$, no extra subscript is used. To derive an expression for α, the following lemmas (proof of Lemma 3 in Section B of the Appendix), will allow us to bound $V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(x_{k+1})$ with a difference between the value functions of problems (2) and (3).

Lemma 3: Assume a solution for Problem 3, with initial point $x_h^*(N-\tilde{N}+1|x_k)$ and prediction horizon $\tilde{N}-1$ or, $V_{\tilde{N}-1}(x_h^*(N-\tilde{N}+1|x_k))$, exists. Then, $\sum_{n=N-\tilde{N}+1}^{N-1} \lambda_{h_2}(n|x_k) = V_{\tilde{N}-1}(x_h^*(N-\tilde{N}+1|x_k))$.

Lemma 4: Consider Lemma 3. For $N \geq 3, 2 \leq \tilde{N} \leq N-1$, we can lower bound $V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(f(x_k, \mu_N^{\tilde{N}}(x_k)))$ as:

$$V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(f(x_k, \mu_N^{\tilde{N}}(x_k))) \geq \lambda_{h_1}(0|x_k) - \Big(V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}+1|x_k)) - V_{\tilde{N}-1}(x_h^*(N-\tilde{N}+1|x_k))\Big). \quad (15)$$

Proof in Appendix, Section B. Using Lemmas 3, 4, and having (7) as a goal, we study the following inequality chain:

$$V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(f(x_k, \mu_N^{\tilde{N}}(x_k))) \geq \lambda_{h_1}(0|x_k) - \Big(V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}+1|x_k)) - V_{\tilde{N}-1}(x_h^*(N-\tilde{N}+1|x_k))\Big) \geq \alpha\lambda_{h_1}(0|x_k). \quad (16)$$

Note that an applicable $\alpha \geq 0$ is only possible if, for any $x_k \in \mathcal{X}_1$, we have

$$\lambda_{h_1}(0|x_k) - \Big(V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}+1|x_k)) - V_{\tilde{N}-1}(x_h^*(N-\tilde{N}+1|x_k))\Big) \geq 0. \quad (17)$$

As in other papers estimating sub-optimality via RDP (e.g.: [6], [7]), a non-applicable value $\alpha < 0$ may arise. Beyond applicability of (12a), $\alpha \geq 0$ is a stability certificate for the system [6]. Later in the section, we will propose a constructive manner to produce $\alpha \geq 0$. The first inequality in (16) always holds and, in case (17) holds, given the positive definiteness of $l(\cdot,\cdot)$, there will always exist $\alpha \in [0,1]$ fulfilling (16). As such, we automatically fulfill the conditions under which Lemma 1 holds. Visual explanation of the upper-bounds previously generated is shown in Figure 2. Based on (16), we can

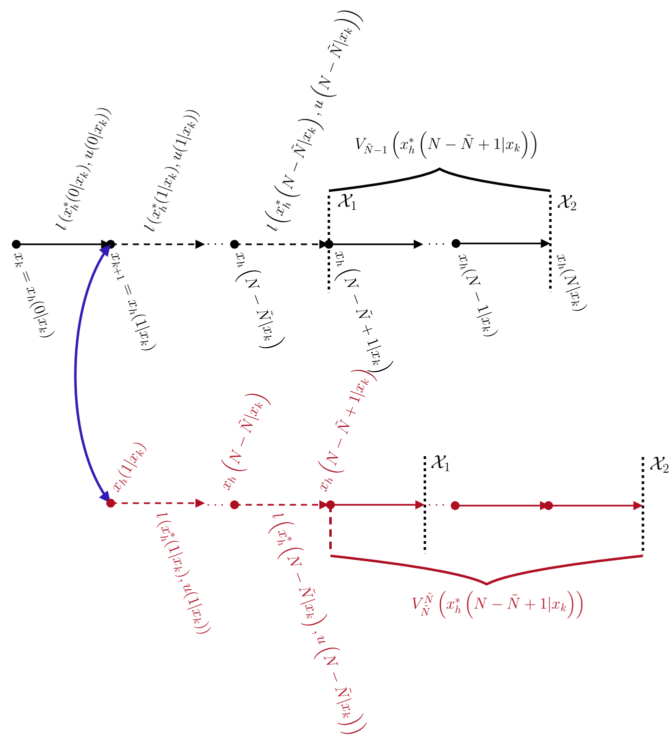


Fig. 2. Black: $V_N^{\tilde{N}}(x_k)$ is equal to the sum of running costs $l(\cdot,\cdot)$ until the state $x_s = x_h(N-\tilde{N}+1|x_k)$, (last one in $\mathcal{X}_1$ - vertical dotted line). From x_s, value function's "tail" is computed by $V_{\tilde{N}-1}(x_s)$. States are shown below the nodes. Blue: Double headed arrow shows $x_h(1|x_k)$ in black and red are the same state. Red: An upper-bound of $V_N^{\tilde{N}}(x_{k+1})$ is derived using $x_{k+1} = x_h(1|x_k)$, built by the sum of running costs $l(\cdot,\cdot)$ until the state $x_s = x_h(N-\tilde{N}+1|x_k)$, which is the penultimate state required to be in $\mathcal{X}_1$. As the starting state now is $x_h(1|x_k)$ and the constraint horizon associated with $\mathcal{X}_1$ is also $N-\tilde{N}$, the state $x_h(1|x_s)$ will be the last required to be in $\mathcal{X}_1$. The "tail" of the value function can be computed by $V_{\tilde{N}}^{\tilde{N}}(x_s)$. Dashed black/red arrows have identical costs and cancel in $V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(x_{k+1})$.

calculate α in different ways. For instance, by computing $V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}+1|x_k)) - V_{\tilde{N}-1}(x_h^*(N-\tilde{N}+1|x_k))$, which according to Lemma 3 can be calculated via the solution of (2) online for $V_N^{\tilde{N}}(x_k)$ and $V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}+1|x_k))$, α can be found, for $x_k \neq 0$, as:

$$\alpha = 1 - \frac{V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}+1|x_k)) - V_{\tilde{N}-1}(x_h^*(N-\tilde{N}+1|x_k))}{\lambda_{h_1}(0|x_k)}. \quad (18)$$

If applicable, (18) estimates the closed loop sub-optimality, but does not give any explicit information on how N and $\tilde{N}$ affect the estimate. Thus, we go on to characterize α explicitly as a function of these parameters. From now on, to ease notation we use $x_s = x_h^*(N-\tilde{N}+1|x_k)$ and $\lambda_0 = \lambda_{h_1}(0|x_k)$.

B. Explicit calculation of α

We rewrite $V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}+1|x_k)) - V_{\tilde{N}-1}(x_h^*(N-\tilde{N}+1|x_k))$ in (16) as:

$$\begin{aligned}
&V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}+1|x_k)) - V_{\tilde{N}-1}(x_h^*(N-\tilde{N}+1|x_k)) \\
&= V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}+1|x_k)) - V_{\tilde{N}}(x_h^*(N-\tilde{N}+1|x_k)) \\
&+ V_{\tilde{N}}(x_h^*(N-\tilde{N}+1|x_k)) - V_{\tilde{N}-1}(x_h^*(N-\tilde{N}+1|x_k)) \\
&= V_{\tilde{N}}^{\tilde{N}}(x_s) - V_{\tilde{N}}(x_s) + V_{\tilde{N}}(x_s) - V_{\tilde{N}-1}(x_s). \qquad (19)
\end{aligned}$$

The first difference after the equality is the cost gap between (2) and (3) at prediction horizon $\tilde{N}$, or the marginal cost of changing the initial constraint. The second is the gap between (3) at prediction horizons $\tilde{N}$ and $\tilde{N}-1$, or the marginal cost of adding one prediction step to a problem with initial prediction horizon $\tilde{N}-1$. We begin with the first difference in (19). Consider Assumption 2; We write $V_{\tilde{N}}^{\tilde{N}}(x_s)$ and $V_{\tilde{N}}(x_s)$ as:

$$V_{\tilde{N}}^{\tilde{N}}(x_s) = \sum_{n=0}^{\tilde{N}-1} l(x_h^*(n|x_s), u_h^*(n|x_s)) \qquad (20a)$$

$$V_{\tilde{N}}(x_s) = \sum_{n=0}^{\tilde{N}-1} l(x_d^*(n|x_s), u_d^*(n|x_s)). \qquad (20b)$$

There always exist $\delta_n \geq 0, n = \{0, \ldots, \tilde{N}-1\}$ such that:

$$\begin{aligned}
&l(x_h^*(n|x_s), u_h^*(n|x_s)) - l(x_d^*(n|x_s), u_d^*(n|x_s)) \leq \\
&\delta_n l(x_d^*(n|x_s), u_d^*(n|x_s)). \qquad (21)
\end{aligned}$$

This inequality help us to bound the cost partially constrained by $\mathcal{X}_1$ by the cost completely constrained by $\mathcal{X}_2$. Observing that, we upper-bound $V_{\tilde{N}}^{\tilde{N}}(x_s) - V_{\tilde{N}}(x_s)$ as:

$$V_{\tilde{N}}^{\tilde{N}}(x_s) - V_{\tilde{N}}(x_s) \leq \sum_{n=0}^{\tilde{N}-1} \delta_n l(x_d^*(n|x_s), u_d^*(n|x_s)). \qquad (22)$$

Although the above is always valid, we limit ourselves to starting state $x_s \neq 0$[2] as $x_s = 0$ would produce $V_{\tilde{N}}^{\tilde{N}}(x_s) - V_{\tilde{N}}(x_s) = 0$. Now, a modified version of [4, Assumption 6.4] is necessary.

Assumption 3 (Maximum rate of cost controllability): Consider the optimal control problems (2) and (3). For any $x_k \in \mathcal{X}_1$, and horizons $N \geq 2, 2 \leq \tilde{N} \leq N$, there exist constants $C_1 \geq 1, C_2 > 0$, decay rates $1 > \sigma_1 > \sigma_2 \geq 0$ and admissible control sequences $u_h(n|x_k), u_d(n|x_s) \in \mathcal{U}$ such that the stage costs $l(\cdot,\cdot)$ along the optimal solutions is asymptotically controllable with rates:

$$\begin{aligned}
&l(x_h^*(n|x_k), u_h^*(n|x_k)) \leq C_1 \sigma_1^n \lambda_0, \qquad (23)\\
&n = \{0, \ldots, N-\tilde{N}\},
\end{aligned}$$

$$\begin{aligned}
&l(x_d^*(n|x_s), u_d^*(n|x_s)) \leq C_2 \sigma_2^{n+1} \lambda_{N-\tilde{N}}, \qquad (24)\\
&n = \{0, \ldots, \tilde{N}-1\},
\end{aligned}$$

where $\lambda_{N-\tilde{N}} = l(x_h^*(N-\tilde{N}|x_k), u_h^*(N-\tilde{N}|x_k))$.

Thus, at its maximum decay values, the part constrained by $\mathcal{X}_1$ attains the origin more slowly than that constrained by $\mathcal{X}_2$ [3]. We now upper bound the second difference in (19) using Assumption 3:

$$\begin{aligned}
&V_{\tilde{N}}(x_s) - V_{\tilde{N}-1}(x_s) = l(x_d^*(\tilde{N}-1|x_s), u_d^*(\tilde{N}-1|x_s)) \\
&\leq C_2 \sigma_2 \sigma_2^{\tilde{N}} \lambda_{N-\tilde{N}}.
\end{aligned}$$

The inequality above is obtained with (24). Via (23) $\lambda_{N-\tilde{N}} \leq C_1 \sigma_1^{N-\tilde{N}} \lambda_0$, implying:

$$V_{\tilde{N}}(x_s) - V_{\tilde{N}-1}(x_s) \leq C_1 C_2 \sigma_1^{N-\tilde{N}} \sigma_2^{\tilde{N}} \lambda_0. \qquad (25)$$

We now put pieces together to claim the following result.

Theorem 1: Consider Assumptions 2 and 3. Then, for $N \geq 3, 2 \leq \tilde{N} \leq N-1$ and for all $x_k \in \mathcal{X}_1$, an explicit expression for α serving as a bound in (8) is:

$$\alpha = 1 - C_1 C_2 \sigma_1^{N-\tilde{N}} \sigma_2 \left(\delta \frac{1-\sigma_2^{\tilde{N}}}{1-\sigma_2} + \sigma_2^{\tilde{N}-1} \right), \qquad (26)$$

where $\delta = \max\{\delta_0, \ldots, \delta_{\tilde{N}-1}\}$.

Proof: We start by simplifying (22), choosing $\delta = \max\{\delta_0, \ldots, \delta_{\tilde{N}-1}\}$:

$$V_{\tilde{N}}^{\tilde{N}}(x_s) - V_{\tilde{N}}(x_s) \leq \delta \sum_{n=0}^{\tilde{N}-1} l(x_d^*(n|x_s), u_d^*(n|x_s))$$

Using (24) on the expression above we obtain:

$$V_{\tilde{N}}^{\tilde{N}}(x_s) - V_{\tilde{N}}(x_s) \leq \sigma_2 C_2 \lambda_{N-\tilde{N}} \left(\delta \sum_{n=0}^{\tilde{N}-1} \sigma_2^n \right),$$

$$V_{\tilde{N}}^{\tilde{N}}(x_s) - V_{\tilde{N}}(x_s) \leq \sigma_2 C_2 \left(\delta \frac{1-\sigma_2^{\tilde{N}}}{1-\sigma_2} \right) \lambda_{N-\tilde{N}}$$

Using (23), we write $\lambda_{N-\tilde{N}}$ as a function of λ_0.

$$\begin{aligned}
&V_{\tilde{N}}^{\tilde{N}}(x_s) - V_{\tilde{N}}(x_s) \leq \sigma_2 C_2 \left(\delta \frac{1-\sigma_2^{\tilde{N}}}{1-\sigma_2} \right) \lambda_{N-\tilde{N}} \\
&\leq C_1 \sigma_1^{N-\tilde{N}} \sigma_2 C_2 \left(\delta \frac{1-\sigma_2^{\tilde{N}}}{1-\sigma_2} \right) \lambda_0
\end{aligned}$$

Including the second difference in (19), using (25), yields:

$$\begin{aligned}
&V_{\tilde{N}}^{\tilde{N}}(x_s) - V_{\tilde{N}-1}(x_s) \\
&= V_{\tilde{N}}^{\tilde{N}}(x_s) - V_{\tilde{N}}(x_s) + V_{\tilde{N}}(x_s) - V_{\tilde{N}-1}(x_s) \\
&\leq C_1 \sigma_1^{N-\tilde{N}} \sigma_2 C_2 \left(\delta \frac{1-\sigma_2^{\tilde{N}}}{1-\sigma_2} \right) \lambda_0 + C_1 C_2 \sigma_1^{N-\tilde{N}} \sigma_2^{\tilde{N}} \lambda_0 \\
&= C_1 \sigma_1^{N-\tilde{N}} \sigma_2 C_2 \left(\delta \frac{1-\sigma_2^{\tilde{N}}}{1-\sigma_2} + \sigma_2^{\tilde{N}-1} \right) \lambda_0
\end{aligned}$$

[2] With Assumption 2 holding from $x_s \neq 0$, the problem is as solvable as (18). There, $V_N^{\tilde{N}}(x_k)$ and $V_{\tilde{N}}^{\tilde{N}}(x_s)$ were assumed to exist. By Lemma 3, existence of $V_N^{\tilde{N}}(x_k)$ implies $V_{\tilde{N}-1}(x_s)$, then $V_{\tilde{N}}(x_s)$ is feasible as the extra state stays in $\mathcal{X}_2$.

[3] Although a special $\mathcal{KL}_0$ function is used here, more geneal ones could also be used. We emphasize that exponential *cost controllability* does not imply *exponential state controllability*. [4, Example 6.5]

Finally, by using (16), and the inequality above we obtain:

$$\begin{aligned} & V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(f(x_k, \mu_N^{\tilde{N}}(x_k))) \\ & \geq \left(1 - C_1\sigma_1^{N-\tilde{N}}\sigma_2 C_2\left(\delta\frac{1-\sigma_2^{\tilde{N}}}{1-\sigma_2} + \sigma_2^{\tilde{N}-1}\right)\right)\lambda_0 \\ & \geq \alpha\lambda_0 \end{aligned} \tag{27}$$

We can then choose:

$$\alpha = 1 - C_1 C_2 \sigma_1^{N-\tilde{N}}\sigma_2\left(\delta\frac{1-\sigma_2^{\tilde{N}}}{1-\sigma_2} + \sigma_2^{\tilde{N}-1}\right)$$

■

Unlike (18), this bound yields a closed-form α as a function of N, $\tilde{N}$, C_1, C_2, σ_1, σ_2, δ. Estimating δ is most demanding since it additionally needs online $V_{\tilde{N}}(x_s)$, beyond $V_{\tilde{N}}^{\tilde{N}}(x_s)$ and $V_N^{\tilde{N}}(x_k)$. In Section VI, we will explain a way to estimate these parameters. If $V_{\tilde{N}}(x_s)$ automatically produces $x_d^*(1|x_s) \in \mathcal{X}_1$, there is no need to calculate $V_{\tilde{N}}^{\tilde{N}}(x_s)$, as $V_{\tilde{N}}^{\tilde{N}}(x_s) = V_{\tilde{N}}(x_s)$, implying δ is not needed, and α in Theorem 1 reduces to:

$$\alpha = 1 - C_1 C_2 \sigma_1^{N-\tilde{N}}\sigma_2^{\tilde{N}}. \tag{28}$$

Proposition 2 (proof in Appendix B) shows how Assumption 3 provides an upper-bound for $J_\infty^{N,\tilde{N}}(x_k)$ (8) based on decay rates and α.

Proposition 2: Consider Assumption 3 and $\alpha \in (0, 1]$. Then by (23), (24) an upper-bound for $J_\infty^{N,\tilde{N}}(x_k)$ is:

$$\begin{aligned} & J_\infty^{N,\tilde{N}}(x_k) \leq C_1\left[\left(\frac{1-\sigma_1^{N-\tilde{N}+1}}{1-\sigma_1}\right) + \right. \\ & \left. C_2\sigma_1^{N-\tilde{N}}\sigma_2\left(\frac{1-\sigma_2^{\tilde{N}-1}}{1-\sigma_2}\right)\right]\frac{\max_{u\in\mathcal{U}} l(x_k, u)}{\alpha}. \end{aligned} \tag{29}$$

Any applicable α works in (29) nonetheless, using Theorem 1 we obtain an upper-bound based on estimated parameters and λ_0 alone. This helps us in one side of our study on the effect of two sets of N and $\tilde{N}$ values on the closed loop upper-bound.

1) Depenence of α on prediction and constraint horizons: As noted, $\alpha \geq 0$ gives an applicable bound and certifies stability. We now study how $N, \tilde{N}$ affect α via Theorem 1.

Proposition 3: Assume $C_1, C_2, \sigma_1, \sigma_2$ and δ are known. Then, via Theorem 1, horizons respecting $N \geq 3, 2 \leq \tilde{N} \leq N-1$ and

$$N - \tilde{N} + 1 \geq \left\lceil \frac{\log\left(C_1 C_2\left[\frac{\delta}{1-\sigma_2} + 1\right]\right)}{\log\left(\frac{1}{\sigma_1}\right)} \right\rceil, \tag{30}$$

guarantee stability in closed loop.

The proof can be found in Appendix B. As per Proposition 3, there exist values for $N, \tilde{N}$ yielding system stability (and thus sub-optimality estimation), given its decay rate characteristics. Next, we discuss methods for determining δ that avoid the explicit online evaluation of $V_{\tilde{N}}^{\tilde{N}}(x_s)$ and $V_{\tilde{N}}(x_s)$.

2) Discussion on δ: To avoid calculation of $V_{\tilde{N}}^{\tilde{N}}(x_s)$ and $V_{\tilde{N}}(x_s)$, δ can be approximated by: $\delta \approx \frac{\sigma_1}{\sigma_2} - 1$. In this heuristic approach the reasoning is, starting from x_s, $V_{\tilde{N}}^{\tilde{N}}(x_s)$'s first term can decay as slow as σ_1 (constraint active), whereas $V_{\tilde{N}}(x_s)$'s first term can decay in the worst case with a σ_2 factor. Despite its simplicity, this is not a bound on δ. We thus derive a conservative yet rigorous upper-bound.

Proposition 4: Assume there exists $\tilde{u}_h(0|x_s) \in \mathcal{U}$ such that $x_s = f(x_s, \tilde{u}_h(0|x_s))$, for all $x_k \in \mathcal{X}_1$ such that $x_s \neq 0$, and $\tilde{u}_h(1|x_s) \in \mathcal{U}$ capable of driving the system from $x_h(0|x_s)$ to $x_d^*(2|x_s)$ in one step. If $V_{\tilde{N}}^{\tilde{N}}(x_s) \neq V_{\tilde{N}}(x_s)$, and there exists $\rho_1, \rho_2 \geq 0$ such that:

$$\rho_1 = \frac{\max_{u\in\mathcal{U}} l(x_h(0|x_s), u)}{\min_{u_1\in\mathcal{U}} l(x_d^*(0|x_s), u_1)} - 1 \tag{31}$$

$$\rho_2 = \frac{\max_{u\in\mathcal{U}} l(x_h(0|x_s), u)}{\min_{u_1,u_2\in\mathcal{U}} l(f(x_d^*(0|x_s), u_1), u_2)} - 1 \tag{32}$$

Then we can choose δ as:

$$\delta = (\rho_1 + \sigma_2\rho_2)\frac{1-\sigma_2}{1-\sigma_2^{\tilde{N}}}. \tag{33}$$

Proof is found in Section B of the Appendix. The input capacity assumption is not overly restrictive, as for a small enough sample time, a small displacement, which in turn requires an arguably small input usage. Despite the looser bound obtained when using expression (33) in (26), it has the advantage of only requiring the computation of $V_N^{\tilde{N}}(x_k)$ for estimating $C_1, C_2, \sigma_1, \sigma_2$ and obtaining x_s, while $V_{\tilde{N}}^{\tilde{N}}(x_s)$ and $V_{\tilde{N}}(x_s)$ are no longer needed in the calculation of δ. Then α becomes computationally simpler compared to (18) and (26). This derivation provides an explicit bound for α based on N, $\tilde{N}$ while giving the designer the option to avoid additional value function evaluations, at the cost of increased conservativeness.

V. Closed loop performance lower bound

A. Calculation of ω as a difference of vale functions

We now revisit the inequality chain (12c) in Proposition 1, as we aim to estimate ω also as a function of $(N, \tilde{N})$. This time, we do so by derivation of an upper bound for $V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(f(x_k, \mu_N^{\tilde{N}}(x_k)))$, and construction of an inequality chain using (12a). We again transform $V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(f(x_k, \mu_N^{\tilde{N}}(x_k)))$ in a difference of value functions, in which the following Lemmas, (proof in Section C of the Appendix) will be useful.

Lemma 5: Assume a solution for $V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}|x_{k+1}))$ exists. Then, $\lambda_{h_1}(N-\tilde{N}|x_{k+1}) + \sum_{n=N-\tilde{N}+1}^{N-1}\lambda_{h_2}(n|x_{k+1}) = V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}|x_{k+1}))$.

Lemma 6: Consider Lemma 5. For $N \geq 3, 2 \leq \tilde{N} \leq N-1$, we can upper-bound $V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(f(x_k, \mu_N^{\tilde{N}}(x_k)))$ as:

$$\begin{aligned} & V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(f(x_k, \mu_N^{\tilde{N}}(x_k))) \leq \lambda_{h_1}(0|x_k) \\ & - \left(V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}|x_{k+1})) - V_{\tilde{N}-1}(x_h^*(N-\tilde{N}|x_{k+1}))\right). \end{aligned} \tag{34}$$

Using Lemmas 5, 6, and having (12a) as a goal, we enforce the inequality chain below (valid for $\alpha \geq 0$):

$$\frac{\alpha}{1-\omega}[V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(f(x_k, \mu_N^{\tilde{N}}(x_k)))] \leq \frac{\alpha}{1-\omega}\left[\lambda_{h_1}(0|x_k) - \left(V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}|x_{k+1})) - V_{\tilde{N}-1}(x_h^*(N-\tilde{N}|x_{k+1}))\right)\right] \leq \alpha\lambda_{h_1}(0|x_k), \tag{35}$$

A meaningful value of ω can be obtained when, for any $x_{k+1} \in \mathcal{X}_1$, it holds that that:

$$\lambda_{h_1}(0|x_k) - \left(V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}|x_{k+1})) - V_{\tilde{N}-1}(x_h^*(N-\tilde{N}|x_{k+1}))\right) \geq 0. \tag{36}$$

We will later provide horizons to produce meaningful values for ω respecting $0 \leq \omega \leq 1-\alpha$. Although $\omega = 0$ produces the same bound in value of the naive bound $V_N^{\tilde{N}}(x_k)$, conceptually it is a certification that under these circumstances, $V_N^{\tilde{N}}(x_k)$ is a reasonable choice and not a "naive pick", which per-se is a valuable information. On the following Lemma (proof in Section C of Appendix), we comment on the existence of $\omega > 0$ generating a tighter bound than $V_N^{\tilde{N}}(x_k)$.

Lemma 7: Consider (36) and $\alpha \in [0,1]$ in Lemma 1. There exists $\omega \in [0,1)$ such that for all $\alpha \in [0,1]$, (35) is satisfied. Furthermore, if $x_h^*(N-1|x_{k+1}) \neq 0$, then a strictly positive $\omega \in (0,1)$ satisfying (35) exists, impliying a tighter (higher) lower-bound than $V_N^{\tilde{N}}(x_k)$ is attainable. In both cases, Proposition 1 is satisfied.

Note that ω is defined independently of α. Nonetheless, ω serving as a lower bound depends on $\alpha \geq 0$ (e.g.: fulfillment of (17) guaranteeing the existence of $\alpha \in [0,1]$). On top of that, if (36) holds, there exist $\omega \in [0,1)$ such that Lemma 7 holds for all $\alpha \in [0,1]$. If the inequality chain in (12c) is invoked, ω must additionally satisfy the necessary compatibility condition $1-\omega \geq \alpha$. Based on Lemma 7 and (35), we can calculate ω in many different ways. For instance, via computation of $V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}|x_{k+1})) - V_{\tilde{N}-1}(x_h^*(N-\tilde{N}|x_{k+1}))$, obtainable via solution of (2) online for $V_N^{\tilde{N}}(x_k)$, $V_N^{\tilde{N}}(x_{k+1})$ and $V_{\tilde{N}-1}(x_h^*(N-\tilde{N}|x_{k+1}))$[4], ω can be found, for $x_k \neq 0$, as:

$$\omega = \frac{V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}|x_{k+1})) - V_{\tilde{N}-1}(x_h^*(N-\tilde{N}|x_{k+1}))}{\lambda_{h_1}(0|x_k)}. \tag{37}$$

Similar to (18), (37) allows us to estimate the closed-loop lower-bound cost. Nonetheless, it does not make explicit how this estimate varies with the parameters N, $\tilde{N}$. As before, we proceed to characterize ω explicitly in terms of these parameters. We henceforth denote $x_p = x_h^*(N-\tilde{N}|x_{k+1})$ and $\lambda_0^+ = \lambda_{h_1}(0|x_{k+1})$.

[4]The calculation of $V_N^{\tilde{N}}(x_k)$ is needed to obtain $\lambda_{h_1}(0|x_k)$. Via Lemma 3 $V_{\tilde{N}-1}(x_h^*(N-\tilde{N}+1|x_{k+1}))$ can be obtained from $V_N^{\tilde{N}}(x_{k+1})$, but $V_{\tilde{N}-1}(x_h^*(N-\tilde{N}|x_{k+1}))$ is not available, requiring its computation.

B. Explicit Estimation of ω

Following (19), we re-write the bound with x_p rather than $x_h^*(N-\tilde{N}+1|x_k)$:

$$V_{\tilde{N}}^{\tilde{N}}(x_p) - V_{\tilde{N}-1}(x_p) = V_{\tilde{N}}^{\tilde{N}}(x_p) - V_{\tilde{N}}(x_p) + V_{\tilde{N}}(x_p) - V_{\tilde{N}-1}(x_p) \tag{38}$$

The same considerations about marginal costs made for (19) are also valid here. Analogously to (22), we now will use an assumption guaranteeing a lower-bound on elements (38).

Assumption 4: Consider Assumption 2, with initial state $x_p \neq 0$. The value functions $V_{\tilde{N}}^{\tilde{N}}(x_p)$ and $V_{\tilde{N}}(x_p)$ can be written as:

$$V_{\tilde{N}}^{\tilde{N}}(x_p) = \sum_{n=0}^{\tilde{N}-1} l(x_h^*(n|x_p), u_h^*(n|x_p)), \tag{39a}$$

$$V_{\tilde{N}}(x_p) = \sum_{n=0}^{\tilde{N}-1} l(x_d^*(n|x_p), u_d^*(n|x_p)). \tag{39b}$$

We assume there exists $\nu_n \geq 0$ such that [5]:

$$l(x_h^*(n|x_p), u_h^*(n|x_p)) - l(x_d^*(n|x_p), u_d^*(n|x_p)) \geq \nu_n l(x_d^*(n|x_p), u_d^*(n|x_p)), n = \{0, \ldots, \tilde{N}-1\} \tag{40}$$

We can then lower-bound $V_{\tilde{N}}^{\tilde{N}}(x_p) - V_{\tilde{N}}(x_p)$ as:

$$V_{\tilde{N}}^{\tilde{N}}(x_p) - V_{\tilde{N}}(x_p) \geq \sum_{n=0}^{\tilde{N}-1} \nu_n l(x_d^*(n|x_p), u_d^*(n|x_p))$$

Now we state an assumption similar to Assumption 3.

Assumption 5 (Minimum rate of cost controlability): Consider optimal control problems (2) and (3). We assume that, for each $x_{k+1} \in \mathcal{X}_1$, there exists admissible control sequences $u_h(n|x_{k+1}), u_d(n|x_p) \in \mathcal{U}$ such that the system is asymptotically controllable with respect to $l(\cdot,\cdot)$ with rates:

$$l(x_h^*(n|x_{k+1}), u_h^*(n|x_{k+1})) \geq C_3\sigma_3^n\lambda_0^+, \quad n = \{0, \ldots, N-\tilde{N}\}, \tag{41}$$

$$l(x_d^*(n|x_p), u_d^*(n|x_p)) \geq C_4\sigma_4^{n+1}\lambda_{N-\tilde{N}}^+, \quad n = \{0, \ldots, \tilde{N}-1\}, \tag{42}$$

where $\lambda_{N-\tilde{N}}^+ = l(x_h^*(N-\tilde{N}|x_{k+1}), u_h^*(N-\tilde{N}|x_{k+1}))$, $C_3 \geq 1, C_4 > 0, \sigma_1 \geq \sigma_3 > \sigma_4 \geq 0$ and, $\sigma_1 > \sigma_2 \geq \sigma_4 \geq 0$.
This means that the constrained part in $\mathcal{X}_1$ has a minimum decay rate that takes longer to decay when compared to the minimum decay rate of the constrained part in $\mathcal{X}_2$. Sequentially applying (42) and (41) (Assumption 5) yields a lower bound for the second difference in (38):

$$V_{\tilde{N}}(x_p) - V_{\tilde{N}-1}(x_p) = l(x_d^*(\tilde{N}-1|x_p), u_d^*(\tilde{N}-1|x_p)) \geq C_3C_4\sigma_3^{N-\tilde{N}}\sigma_4^{\tilde{N}}\lambda_0^+. \tag{43}$$

Theorem 2: Consider Assumptions 2, 4 and 5. Then, for $N \geq 3, 2 \leq \tilde{N} \leq N-1$ and for all $x_k \in \mathcal{X}_1, x_k \neq 0$, an

[5]Different from $\delta_n \geq 0$ satisfying (21), which always exists, here $\nu_n \geq 0$ satisfying (40) may not exist. This is since $l(x_d^*(n|x_p), u_d^*(n|x_p)) \geq l(x_h^*(n|x_p), u_h^*(n|x_p))$ may happen for a given n, underlying the need to assume existence of ν_n.

explicit expression for ω serving as a bound in (12b) can be described by:

$$\omega = C_3 C_4 \kappa \sigma_3^{N-\tilde{N}} \sigma_4 \left(\nu \frac{1-\sigma_4^{\tilde{N}}}{1-\sigma_4} + \sigma_4^{\tilde{N}-1} \right), \quad (44)$$

where $\nu = \min\{\nu_0, \ldots, \nu_{\tilde{N}-1}\}$ and $\kappa \geq 0,\ s.t.,\ \lambda_0^+ \geq \kappa\lambda_0$.

Proof: Consider the inequality:

$$V_{\tilde{N}}^{\tilde{N}}(x_p) - V_{\tilde{N}}(x_p) \geq \sum_{n=0}^{\tilde{N}-1} \nu_n l(x_d^*(n|x_p), u_d^*(n|x_p)).$$

We simplify the expression above by choosing $\nu = \min\{\nu_0, \ldots, \nu_{\tilde{N}-1}\}$. Then:

$$V_{\tilde{N}}^{\tilde{N}}(x_p) - V_{\tilde{N}}(x_p) \geq \nu \sum_{n=0}^{\tilde{N}-1} l(x_d^*(n|x_p), u_d^*(n|x_p)).$$

Using (42) the expression above becomes

$$V_{\tilde{N}}^{\tilde{N}}(x_p) - V_{\tilde{N}}(x_p) \geq \sigma_4 C_4 \lambda_{N-\tilde{N}}^+ \left(\nu \sum_{n=0}^{\tilde{N}-1} \sigma_4^n \right),$$

$$V_{\tilde{N}}^{\tilde{N}}(x_p) - V_{\tilde{N}}(x_p) \geq \sigma_4 C_4 \left(\nu \frac{1-\sigma_4^{\tilde{N}}}{1-\sigma_4} \right) \lambda_{N-\tilde{N}}^+$$

By using (41), we can write $\lambda_{N-\tilde{N}}^+$ as a function of λ_0^+, i.e.,

$$V_{\tilde{N}}^{\tilde{N}}(x_p) - V_{\tilde{N}}(x_p) \geq \sigma_4 C_4 \left(\nu \frac{1-\sigma_4^{\tilde{N}}}{1-\sigma_4} \right) \lambda_{N-\tilde{N}}^+$$
$$\geq C_3 \sigma_3^{N-\tilde{N}} \sigma_4 C_4 \left(\nu \frac{1-\sigma_4^{\tilde{N}}}{1-\sigma_4} \right) \lambda_0^+$$

We now include the contribution of the second difference in the sum (38), with the lower bound calculated in (43), producing:

$$\begin{aligned} &V_{\tilde{N}}^{\tilde{N}}(x_p) - V_{\tilde{N}-1}(x_p) \\ &\geq C_3 \sigma_3^{N-\tilde{N}} \sigma_4 C_4 \left(\nu \frac{1-\sigma_4^{\tilde{N}}}{1-\sigma_4} \right) \lambda_0^+ + C_3 C_4 \sigma_3^{N-\tilde{N}} \sigma_4^{\tilde{N}} \lambda_0^+ \\ &= C_3 \sigma_3^{N-\tilde{N}} \sigma_4 C_4 \left(\nu \frac{1-\sigma_4^{\tilde{N}}}{1-\sigma_4} + \sigma_4^{\tilde{N}-1} \right) \lambda_0^+. \end{aligned}$$

Finally we note that

$$\lambda_0^+ = l(f(x_k, u_h^*(0|x_k)), u_h^*(0|x_{k+1})) \geq l(f(x_k, u_h^*(0|x_k)), 0)$$
$$\implies \lambda_0^+ \geq \kappa\lambda_0, \quad \kappa = \frac{l(f(x_k, u_h^*(0|x_k)), 0)}{l(x_k, u^*(0|x_k))}. \quad (45)$$

We can then choose:

$$\omega = C_3 C_4 \kappa \sigma_3^{N-\tilde{N}} \sigma_4 \left(\nu \frac{1-\sigma_4^{\tilde{N}}}{1-\sigma_4} + \sigma_4^{\tilde{N}-1} \right).$$

■

Unlike (37), the bound above provides a closed-form estimate for ω as a function of the prediction horizon N, the secondary constraint horizon $\tilde{N}$, and the parameters C_3, C_4, σ_3, σ_4, and ν. The "analogous term to δ", which is ν is the most demanding to be estimated as expected. Unlike in (37), it requires the additional online computation of $V_{\tilde{N}}(x_p)$, in addition to the already needed values $V_N^{\tilde{N}}(x_k)$, $V_N^{\tilde{N}}(x_{k+1})$ and $V_{\tilde{N}-1}(x_h^*(N-\tilde{N}|x_{k+1}))$. Analogous to the discussion in the previous section, if the optimal choice automatically leads to $x_d^*(N-\tilde{N}+1|x_{k+1}) \in \mathcal{X}_1$ [6], then the term connected $V_{\tilde{N}}^{\tilde{N}}(x_p) - V_{\tilde{N}}(x_p)$ in Theorem 2 cancels out as $V_{\tilde{N}}^{\tilde{N}}(x_p) = V_{\tilde{N}}(x_p)$. Thus, a calculation for ν is not needed since ω in Theorem 2 reduces to:

$$\omega = C_3 C_4 \kappa \sigma_3^{N-\tilde{N}} \sigma_4^{\tilde{N}}. \quad (46)$$

Using Theorem 2 with Assumptions 4 and 5, we obtain in Proposition 5, a lower bound for $J_\infty^{N,\tilde{N}}(x_k)$ depending solely on the estimated parameters and λ_0.

Proposition 5: Consider Assumptions 4 and 5. Then by (41) and (42), the lower-bound for $J_\infty^{N,\tilde{N}}(x_k)$ can be written as:

$$\begin{aligned} J_\infty^{N,\tilde{N}}(x_k) \geq C_3\kappa \Bigg[&\left(\frac{1-\sigma_3^{N-\tilde{N}+1}}{1-\sigma_3} \right) + \\ &C_4 \sigma_3^{N-\tilde{N}} \sigma_4 \left(\frac{1-\sigma_4^{\tilde{N}-1}}{1-\sigma_4} \right) \Bigg] \frac{\min_{u\in\mathcal{U}} l(x_k, u)}{1-\omega}. \end{aligned} \quad (47)$$

From (47) and the positive definiteness of $l(\cdot,\cdot)$, $\min_{u\in\mathcal{U}} l(x_k, u) = 0$ only if $x_k = 0$. Any estimated ω can be used on Proposition 5, nonetheless, using Theorem 2, provides a lower-bound based on estimated parameters and λ_0 alone. Having upper and lower bounds of $J_\infty^{N,\tilde{N}}(x_k)$, we can address the other main goal, comparing the effect of different $\tilde{N}$ on closed loop trajectories of a nonlinear system controlled by (2). Namely, if design options $\tilde{N}_1 \leq \tilde{N}_2$, could produce:

$$J_\infty^{N,\tilde{N}_2}(x_k) \geq J_\infty^{N,\tilde{N}_1}(x_k), \quad (48)$$

implying $\tilde{N}_2$ is decidedly a worse choice. A sufficient condition producing (48) is, when applying (12c), the lower-bound produced by $\tilde{N}_2$ exceeds the upper-bound of $\tilde{N}_1$:

$$\frac{V_N^{\tilde{N}_1}(x_k)}{\alpha_{\tilde{N}_1}} \leq \frac{V_N^{\tilde{N}_2}(x_k)}{1-\omega_{\tilde{N}_2}}.$$

Using Propositions 2 and 5, the above expression can be tested as a function of decay rates, prediction and constraint horizons.

1) Dependence of ω on prediction and constraint horizons: We study parameter–horizon choices that yield meaningful relations for ω, using Theorem 2.

Proposition 6: Consider Proposition 3, and that $C_3, C_4, \kappa, \sigma_3$ and σ_4 is known. If $\sigma_4, \kappa > 0$, $\delta \geq \nu$ and horizons respecting $N \geq 3, 2 \leq \tilde{N} \leq N-1$ and

$$N - \tilde{N} \geq \left\lceil \frac{\log\left(\frac{C_3 C_4 \kappa}{C_1 C_2}\right)}{\log\left(\frac{\sigma_1}{\sigma_3}\right)} \right\rceil, \quad (49)$$

or, if $\sigma_4 = 0$ or $\kappa = 0$ (without any extra requirement) then, $0 \leq \omega \leq 1-\alpha$ will be respected.

The proof can be found in Appendix C. Propositions 3 and 6 together generate conditions in which nontrivial upper and lower bound exist.

[6] Note we use the dependency on x_{k+1} and not x_k, reason why we use $N - \tilde{N} + 1$.

2) Discussion on ν: The simplest way to obtain a value for ν, in an heuristic fashion, without the need to calculate $V_{\tilde{N}}(x_p)$ (we still need to obtain $V_N^{\tilde{N}}(x_k)$, $V_N^{\tilde{N}}(x_{k+1})$ and get $V_{\tilde{N}}^{\tilde{N}}(x_p)$ via Lemma 7) is to approximate it by: $\nu \approx \frac{\sigma_3}{\sigma_4} - 1$. The rationale is that, starting from the same point, $V_{\tilde{N}}^{\tilde{N}}(x_p)$ will have its first term decaying as fast as σ_3 (active constraint), whereas the first term of $V_{\tilde{N}}(x_p)$ will decay with a σ_4 factor. A lower bound on ν, proposed below (proof in Section C of the Appendix.):

Proposition 7: Let $V_{\tilde{N}}^{\tilde{N}}(x_p) \neq V_{\tilde{N}}(x_p)$. Furthermore assume that there exists $\tilde{u}_d(0|x_p) \in \mathcal{U}$ such that $x_p = f(x_p, \tilde{u}_d(0|x_p))$, for all $x_{k+1} \in \mathcal{X}_1$ such that $x_p \neq 0$, and $\tilde{u}_d(1|x_p) \in \mathcal{U}$ capable of driving the system from $x_d(0|x_p)$ to $x_h^*(2|x_p)$ in one step. If there exists $\phi_1, \phi_2 \geq 0$ such that:

$$\phi_1 = 1 - \frac{\max_{u \in \mathcal{U}} l(x_d(0|x_p), u)}{\min_{u_1 \in \mathcal{U}} l(x_h^*(0|x_p), u_1)} \quad (50)$$

$$\phi_2 = 1 - \frac{\max_{u \in \mathcal{U}} l(x_d(0|x_p), u)}{\min_{u_1, u_2 \in \mathcal{U}} l(f(x_h^*(0|x_p), u_1), u_2)} \quad (51)$$

$$(52)$$

Then we can choose ν as:

$$\nu = \frac{\left(\phi_1 + C_4 \sigma_4^2 \phi_2\right)}{\sigma_4} \frac{1 - \sigma_4}{1 - \sigma_4^{\tilde{N}}}. \quad (53)$$

Considerations regarding input capacity assumption mildness used for Proposition 4, are also valid here. Despite the looser bound obtained when using (53) in (44), it has the advantage of only requiring the computation of $V_N^{\tilde{N}}(x_{k+1})$ for estimating $C_3, C_4, \sigma_3, \sigma_4$ and obtaining x_p and $V_N^{\tilde{N}}(x_k)$ for calculation of λ_0 while $V_{\tilde{N}}^{\tilde{N}}(x_p)$ and $V_{\tilde{N}}(x_p)$ are no longer needed for obtaining ν, simplifying ω's computation compared to (37) and (44). This result gives an explicit bound for ω based on N, $\tilde{N}$, allowing the designer to avoid extra value function evaluations, at the cost of increased conservativeness.

VI. Numerical simulations

A. Nonlinear example

We study the derived bounds on a 6-DoF nonlinear quadrotor model. We adopt the North East Down (NED) orientation, with the inertial frame located on earth's surface, and O_{xyz} fixed at the quadrotor centre of mass, as in Figure 3. We

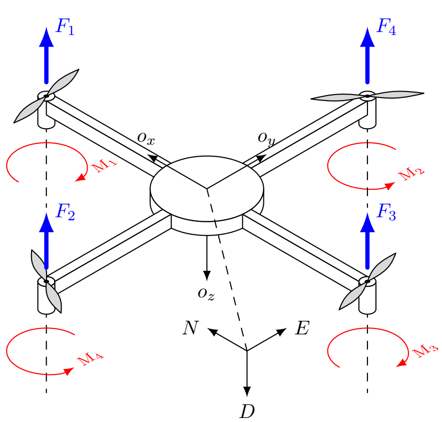


Fig. 3. Reference frames for UAV [33].

model the quadrotor using a simplified nonlinear model valid for small angle deviations [34]:

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}),$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \\ u \\ v \\ w \\ p \\ q \\ r \end{bmatrix}, f(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} w(\phi\psi + \theta) - v(\psi - \phi\theta) + u \\ v(1 + \phi\psi\theta) - w(\phi - \psi\theta) + u\psi \\ w - u\theta + v\phi \\ p + r\theta + q\phi\theta \\ q - r\phi \\ r + q\phi \\ rv - qw - g\theta \\ pw - ru + g\phi \\ qu - pv + g - \dfrac{f_t}{m} \\ \dfrac{I_y - I_z}{I_x} rq + \dfrac{\tau_x}{I_x} \\ \dfrac{I_z - I_x}{I_y} pr + \dfrac{\tau_y}{I_y} \\ \dfrac{I_x - I_y}{I_z} pq + \dfrac{\tau_z}{I_z} \end{bmatrix}.$$

In the above equation $\mathbf{x} \in \mathbb{R}^{12}$ is the vector composed of: positions x, y, z, velocities u, v, w, angular displacements ϕ, θ, ψ and angular velocities p, q, r in/around the directions o_x, o_y, o_z, with respect to the inertial frame [34]. The angular movement around the o_x, o_y, o_z axis are often also called roll, pitch and yaw. The input vector $\mathbf{u} = [f_t, \tau_x, \tau_y, \tau_z]^T$, is composed by $f_t = F_1 + F_2 + F_3 + F_4$, the thrust force generated by propellers and τ_x, τ_y, τ_z are torques generated along the o_x, o_y, o_z directions. Finally, I_x, I_y, I_z are inertia matrix diagonal components (off-diagonal terms neglected). UAV$_1$ has its initial state $\mathbf{x}_{\text{init}} = [1, 2, -1, 0_{1x9}]^T$, and UAV$_2$ has its state fixed to $\mathbf{x}_{\text{o}} = [0.4, 1.5, -0.2, 0_{1x9}]^T$. With respect to the NED orientation, in which a negative Z means displacement above ground, UAV$_1$ starts hovering at $[1, 2, 1]^\top$ and is to land at the origin while avoiding UAV$_2$, hovering at $[0.4, 1.5, 0.2]^\top$. We discretize the system with sample time $h = 0.4s$ and solve (2). The input constraint set is $\mathcal{U} = \{\mathbf{u}(n|\mathbf{x}_k) \in \mathbb{R}^4 | -u_{max} \leq \mathbf{u}(n|\mathbf{x}_k) \leq u_{max}, \forall n = 0, \ldots, N-1\}$ and the state constraints set $\mathcal{X}_1 = \mathcal{X}$, defined for all $n = 0, \ldots, N - \tilde{N}$ as in (54) and $\mathcal{X}_2 = \mathbb{R}^{12}$.

$$\mathcal{X} = \left\{ \mathbf{x}(n+1|\mathbf{x}_k) \in \mathbb{R}^{12} \;\middle|\; \begin{array}{l} \|d_{1,2}(n+1|\mathbf{x}_k)\|_2^2 \geq r_{obs}^2 + \epsilon, \\ d_{1,2}(n+1|\mathbf{x}_k) = \\ [x(n+1|\mathbf{x}_k) - x_o, \\ y(n+1|\mathbf{x}_k) - y_o, \\ z(n+1|\mathbf{x}_k) - z_o]^\top, \\ z(n+1|\mathbf{x}_k) \leq 0, \\ |\phi(n+1|\mathbf{x}_k)|, |\theta(n+1|\mathbf{x}_k)|, \\ |\psi(n+1|\mathbf{x}_k)| \leq \dfrac{\pi}{9}, \\ \|v_{1,2}(n+1|\mathbf{x}_k)\|_2 \leq 2, \\ v_{1,2}(n+1|\mathbf{x}_k) = [u(n+1|\mathbf{x}_k), \\ v(n+1|\mathbf{x}_k), w(n+1|\mathbf{x}_k)]^\top, \\ |p(n+1|\mathbf{x}_k)|, |q(n+1|\mathbf{x}_k)|, \\ |r(n+1|\mathbf{x}_k)| \leq \dfrac{\pi}{18} \end{array} \right\} \quad (54)$$

Under (54), we enforce for UAV$_1$: The usual distance constraint $\|d_{1,2}(n+1|\mathbf{x}_k)\|_2^2 \geq r_{\text{obs}}^2 + \epsilon$ where $r_{obs} = 0.5$ defines the safety distance around UAV_2 and $\epsilon > 0$ (here chosen $\epsilon = 0.1$) is a small tolerance value, the ground avoidance constraint $z(n+1|\mathbf{x}_k) \leq 0$ (having $z \geq 0$ implies UAV$_1$ would crash into the ground), attitude bounds $|\phi(n+1|\mathbf{x}_k)|, |\theta(n+1|\mathbf{x}_k)|, |\psi(n+1|\mathbf{x}_k)| \leq \frac{\pi}{9}$ introduced due to the model being valid for small angles only (threshold assumed to be $20°$), velocity constraints $\|v_{1,2}(n+1|\mathbf{x}_k)\|_2 \leq 2$ limited to be at most $2m/s$, and attitude rate constraint $|p(n+1|\mathbf{x}_k)|, |q(n+1|\mathbf{x}_k)|, |r(n+1|\mathbf{x}_k)| \leq \frac{\pi}{18}$ limited to be smaller than $10°$ per sample. The stage cost is $l(x(n|x_k), u(n|x_k)) = x(n|x_k)^\top Q x(n|x_k) + u(n|x_k)^\top R u(n|x_k)$. Figure 4 shows UAV_1's trajectory obtained by solving (2) with $N = 16$ and $N - \tilde{N} = 3$. The safety is verified by checking the minimum distance value obtained which is of $0.59m$, being thus greater than r_{obs}. We now estimate $J_\infty^{N,\tilde{N}}(x_k)$ via (8) and

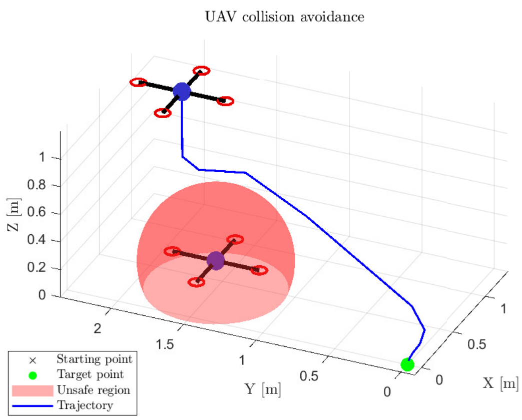


Fig. 4. Trajectory of UAV 1 avoiding collision with UAV 2.

(12b) while using Theorems 1 and 2 to obtain values for α and ω respectively. We compute σ_1, σ_2, and δ using the open loop value function $V_N^{\tilde{N}}(x_0)$. We estimate $\sigma_1 = \sigma_1(x_0) = \max(\sigma_h(1|x_0), \ldots, \sigma_h(N-\tilde{N}|x_0))$ and $\sigma_2 = \sigma_2(x_0) = \max(\sigma_d(N-\tilde{N}+1|x_0), \ldots, \sigma_d(N-1|x_0))$, where $\sigma_h(n|x_k)$ and $\sigma_d(n|x_k)$ are calculated, based on Assumption 3, as:

$$\sigma_h(n|x_k) = f_1(n, x_k, \lambda_0),\ n = 1, \ldots, N-\tilde{N}, \tag{55}$$

$$\sigma_d(n|x_k) = f_2(n, x_k, \lambda_{N-\tilde{N}}),\ n = N-\tilde{N}+1, \ldots, N-1, \tag{56}$$

$$f_1(n,a,b) = \left(\frac{l(x_h^*(n|a), u_h^*(n|a))}{b}\right)^{\frac{1}{n}},$$

$$f_2(n,a,b) = \left(\frac{l(x_h^*(n|a), u_h^*(n|a))}{b}\right)^{\frac{1}{n+1-(N-\tilde{N})}}.$$

$C_1 = C_2 = 1$ due the estimation of $\sigma_h(n|x_k)$ and $\sigma_d(n|x_k)$.

Similarly, σ_3, σ_4, and ν are calculated with $V_N^{\tilde{N}}(x_1)$. We estimate $\sigma_3 = \sigma_3(x_1) = \min(\sigma_h^+(1|x_1), \ldots, \sigma_h^+(N-\tilde{N}|x_1))$, and $\sigma_4 = \sigma_4(x_1) = \min(\sigma_d^+(N-\tilde{N}+1|x_1), \ldots, \sigma_d^+(N-1|x_1))$, where $\sigma_h^+(n|x_{k+1})$ and $\sigma_d^+(n|x_{k+1})$ are calculated, based on Assumption 5, as:

$$\sigma_h^+(n|x_{k+1}) = f_1(n, x_{k+1}, \lambda_0^+),\ n = 1, \ldots N-\tilde{N}, \tag{57}$$

$$\sigma_d^+(n|x_{k+1}) = f_2(n, x_{k+1}, \lambda_{N-\tilde{N}}^+), \tag{58}$$
$$n = N-\tilde{N}+1, \ldots, N-1.$$

We also set $C_3 = C_4 = 1$ as before due to the estimation of $\sigma_h^+(n|x_{k+1})$ and $\sigma_d^+(n|x_{k+1})$. For all σ's we use $l(x_h^*(n|x_{id}), u_h^*(n|x_{id})), id = \{k, k+1\}$ so that we can calculate only $V_N^{\tilde{N}}(x_{id})$ and do not need to calculate the unconstrained equivalent.

Calculation of δ and ν are done via the bounds discussed in (33) and (53), and via $\delta \approx \frac{\sigma_1}{\sigma_2} - 1$ and $\nu \approx \frac{\sigma_3}{\sigma_4} - 1$ respectively. Results are compiled in Table I for the horizon pairs $N = 16$, $N - \tilde{N} = 3$ and $N = 27$, $N - \tilde{N} = 3$.

Horizons	$J_\infty^{N,\tilde{N}}(x_0)$	δ and ν	$\frac{V_N^{\tilde{N}}(x_0)}{\alpha}$	$\frac{V_N^{\tilde{N}}(x_0)}{1-\omega}$
$N=16$ $N-\tilde{N}=3$	475.09	(33) and (53) Approx.	1310.44 842.89	442.77 N.A.
$N=27$ $N-\tilde{N}=3$	475.10	(33) and (53) Approx.	835.65 630.03	442.78 N.A.

TABLE I
BOUNDS BY HORIZONS AND ESTIMATION METHODS.

The lower bound results using (53) are close to the actual closed loop cost. In both cases the approximation $\nu \approx \frac{\sigma_3}{\sigma_4} - 1$ obtained was not applicable, since $\omega > 1 - \alpha$. Upper bounds calculated using both (33) and the approximation $\delta \approx \frac{\sigma_1}{\sigma_2} - 1$ give valid, albeit looser bounds, with the latter providing less conservative results. During certain simulation instances, (33) yielded low α values, which despite producing a very loose upper bound for the closed loop, still provided a valuable information as the region in which $\alpha > 0$ implies closed loop stability on this region. We underline that the calculations use only the state x_0 and as such are still an approximation.

B. Linear system comparison

We now compare the bounds in Theorem 1 with the ones in [27], which also obtain (8) via RDP, but α is estimated as:

$$\alpha = 1 - \frac{\beta^{N-\tilde{N}+1}}{(\beta+1)^{N-\tilde{N}-1}}, \tag{59}$$

and $\beta > 0$ is obtained for $n \in \{\tilde{N}+1, \ldots, N\}$ as:

$$V_{\tilde{N}+1}^{\tilde{N}}(x_k) \leq (\beta+1) V_{\tilde{N}}^{\tilde{N}}(x_k), \tag{60a}$$

$$V_n^{\tilde{N}}(x_k) \leq (\beta+1) l(x_k, u_h^*(N-n|x_k)). \tag{60b}$$

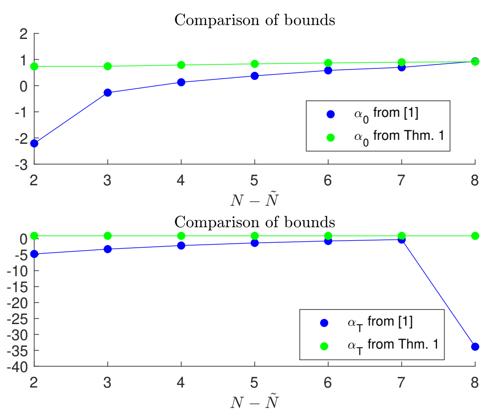
Following the example in [27], consider the linear double integrator $x(i+1|k) = Ax(i|k) + Bu(i|k)$ with $x(i|k) = [p_x(i|k), p_y(i|k), v_x(i|k), v_y(i|k)]^T$ listing positions and velocities, and $u(i|k) = [a_x(i|k), a_y(i|k)]^T$ containing the accelerations, both in the x, y coordinates. We use a quadratic cost $l(x(i|k), u(i|k)) = x(i|k)^T Q x(i|k) + u(i|k)^T R u(i|k)$ and input bounds $\mathcal{U} = \{u(i|k) \in \mathbb{R}^2, s.t. \ -2 \leq u_r(i|k) \leq 2\}$, where $r = \{1, 2\}$ are the rows of $u(i|k)$, and $i = 0, \ldots, N-1$. State constraints are: velocity bounds $|v_x(j+1|k)| + |v_y(j+1|k)| \leq 2$ and for position bounds, we re-utilize the CBF *candidate functions*

$$h_1(x(j|k)) = \frac{5}{9} p_x(j|k) + p_y(j|k) + \frac{0.5}{9}, \tag{61a}$$

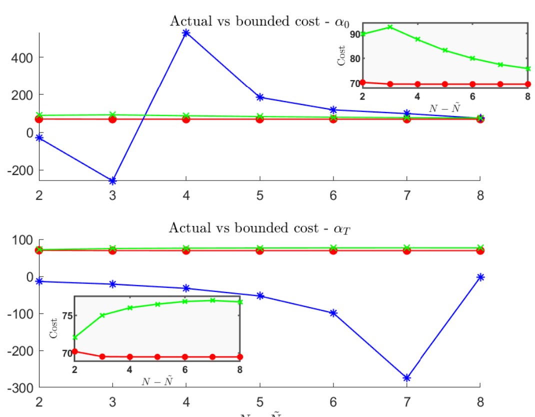
$$h_2(x(j|k)) = p_x(j|k) - p_y(j|k) + 1.6, \tag{61b}$$

The system is subject to velocity and positions (CBF) constraints $h_1(x(j+1|k)) \geq (1-\gamma) h_1(x(j|k))$ and $h_2(x(j+$

$1|k)) \geq (1-\gamma)h_2(x(j|k))$, with $\gamma = 0.8$ for $j = 0, \dots, N-\tilde{N}$. The initial state is set to $x_0 = [-0.8, 0.6, -0.45, 0.65]$, and the target state is the origin. Here we compute α in two ways, both using $\delta \approx \frac{\sigma_1}{\sigma_2} - 1$. The difference is if (55) and (56) are estimated with x_0 alone or the full trajectory x_k, $k = 0, \dots, T$ (T samples). Using x_0 only, yields a "local" α and requires computing β from [27] using only x_0 as well, while using the full trajectory produces more conservative results, but matches the original setting of [27]. For comparison we solve (2) repeatedly for $N = 10$ and $N = 20$ while varying $\tilde{N}$. For each $\tilde{N}$ and a fixed N, we compute $\sigma_{1,0} = \sigma_1(x_0)$, $\sigma_{1,T} = \max(\sigma_1(x_0), \dots, \sigma_1(x_T))$, $\sigma_{2,0} = \sigma_2(x_0)$, $\sigma_{2,T} = \max(\sigma_2(x_0), \dots, \sigma_2(x_T))$, where each $\sigma_1(x_k), \sigma_2(x_k)$ is again obtained via (55) and (56). We set $C_1 = C_2 = 1$ and from now on we use α_0 and α_T for the estimations using x_0 only, and the full trajectory respectively. Using $\delta \approx \frac{\sigma_1}{\sigma_2} - 1$ (rather than (33)) is simpler and performed better: (33) produced more cases with $\alpha < 0$, making the bounds inapplicable. As a benchmark, we computed β_0 from x_0 and β_T from the full trajectory to obtain α_0 and α_T according to [27]. For $N = 10$, see Fig. 5a ($\alpha' s$) and Fig. 5b (closed-loop cost).



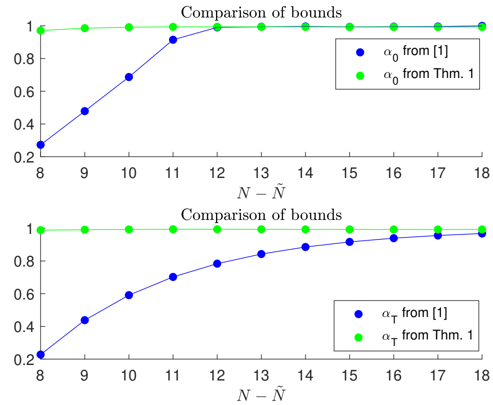
(a) For $N = 10$, α_0 and α_T vs. $N - \tilde{N}$.



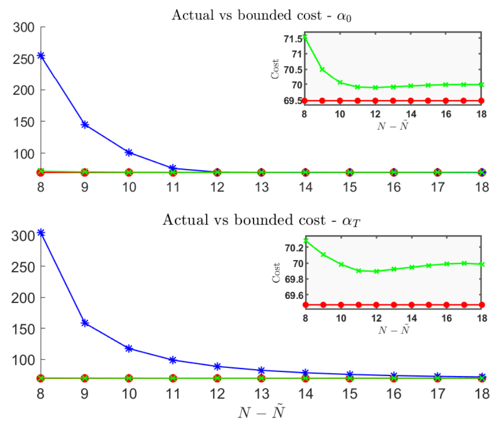
(b) Closed-loop cost for $N = 10$ vs. $N - \tilde{N}$. Blue/green: estimates from [27] and (26); red: actual.

Fig. 5. $\boldsymbol{\alpha}$'s and closed-loop cost for $\boldsymbol{N = 10}$ as a function of $\boldsymbol{N - \tilde{N}}$.

The α estimate from Theorem 1 appears to converge faster than the one in (59), and hold over a larger region, whether using x_0 or the full trajectory. In Figure 5a, (59) yields a larger negative α region (inapplicable bound), whereas Theorem 1 gives $\alpha \in (0, 1)$. Figure 6a shows α for $N = 20$, and Figure 6b the corresponding closed-loop estimation. From



(a) For $N = 20$, α_0 and α_T vs. $N - \tilde{N}$.



(b) Closed-loop cost for $N = 20$ vs. $N - \tilde{N}$. Blue/green: estimates from [27] and (26); red: actual.

Fig. 6. $\boldsymbol{\alpha}$'s and closed-loop cost for $\boldsymbol{N = 20}$ as a function of $\boldsymbol{N - \tilde{N}}$.

Figures 6a and 6b, we see that Theorem 1 produces the closest value to $J_\infty^{N,\tilde{N}}(x_0)$ when $N - \tilde{N} = 12$. Increasing $N - \tilde{N}$ beyond 12 raises the $\frac{V_N^{\tilde{N}}(x_0)}{\alpha_0}$ by ($< 0.01\%$ per unit increase in $N - \tilde{N}$) and produces $\sigma_{2,0} > \sigma_{1,0}$ and $\sigma_{2,T} > \sigma_{1,T}$. As $N - \tilde{N} \to 20$ $\sigma_{2,0}$ and $\sigma_{2,T}$ tend to 1. Based on the calculation of (55) and (56), it is expected that close to the equilibrium $\sigma_{1,0}, \sigma_{1,T}, \sigma_{2,0}, \sigma_{2,T} \to 1$, possibly implying that we are beyond the transient point, relevant to calculate system convergence's speed. As such, estimation of (55) and (56) become less reliable, but still conveys relevant information. Finally, for high values of N and $N - \tilde{N}$ (e.g.: $N = 20$ and $N - \tilde{N} = 14$), α_0 obtained from (59) is around 0.3% higher than α_0 from Theorem 1, which is not seen for α_T, where Theorem 1 always outperforms (59). This may indicate a complex relation between these results, which ought to be explored further.

VII. Conclusion

We presented an MPC formulation with two constraint types. The first is a control-invariant set which presents higher maximum and minimum decay rate values (slower decay) and ensures recursive feasibility. The second is a standard state constraint set, which contains the first set, possibly with lower decay-rate bounds, encompassing a partially constrained formulation. Our contributions are: 1) a conceptual generalization of closed-loop MPC estimation; 2) a tighter (for short horizons) computation of closed-loop upper bounds from open-loop value functions using decay rates and constraint horizons; and 3) a new and less expensive closed-loop lower-bound than the usual infinite-horizon open-loop value. We explored different parameter estimation and bounding techniques and validate the theory on nonlinear and linear examples, showing larger validity regions than existing results. Future work will address offline parameter estimation, improve closed-loop comparisons across horizons, and deepen understanding of this bounds with existing ones.

APPENDIX

A. Relaxed Dynamic Programming Proofs

1) Proof of Lemma 2: We reconsider the second inequality of (10) (also valid for the first inequality of (64)) rearranged:

$$V_N^{\tilde{N}}(x_k) - \alpha \sum_{j=0}^{M} l(x_{k+j}, \mu_N^{\tilde{N}}(x_{k+j})) \geq V_N^{\tilde{N}}(x_{k+1+M}) \geq 0.$$

We want to study its behavior when $M \to \infty$:

$$V_N^{\tilde{N}}(x_k) - \lim_{M\to\infty} \alpha \sum_{j=0}^{M} l(x_{k+j}, \mu_N^{\tilde{N}}(x_{k+j})) \geq \lim_{M\to\infty} V_N^{\tilde{N}}(x_{k+1+M}) \geq 0. \tag{62}$$

The first inequality above can further be rearranged as:

$$V_N^{\tilde{N}}(x_k) \geq \lim_{M\to\infty} \alpha \sum_{j=0}^{M} l(x_{k+j}, \mu_N^{\tilde{N}}(x_{k+j})) \geq 0. \tag{63}$$

As $V_N^{\tilde{N}}(x_k)$ is finite, so must the infinite sum be. As $l(\cdot,\cdot) > 0$ and $\alpha \in [0,1]$ elements of the sum in itself must go to zero, meaning $l(x_{k+1+M}, \mu_N^{\tilde{N}}(x_{k+1+M})) \to 0$ as $M \to \infty$. Running cost positive definiteness ensures, $x_{k+1+M} \to 0$ as $M \to \infty$ and thus, $V_N^{\tilde{N}}(x_{k+1+M}) \to 0$ as $M \to \infty$.

2) Proof of Proposition 1: By performing the same steps from (9) to (10), we get the following bounds:

$$\begin{aligned} &V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(x_{k+1+M}) \\ &\geq \alpha \sum_{j=0}^{M} l(x_{k+j}, \mu_N^{\tilde{N}}(x_{k+j})) := \alpha J_M^{N,\tilde{N}}(x_k) \\ &\geq \frac{\alpha}{1-\omega}\left[V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(x_{k+1+M})\right]. \end{aligned} \tag{64}$$

As $M \to \infty$ (64) simplifies (Lemma 2). Applying Lemma 2 on (64) produces

$$V_N^{\tilde{N}}(x_k) \geq \alpha J_\infty^{N,\tilde{N}}(x_k) \geq \frac{\alpha}{1-\omega} V_N^{\tilde{N}}(x_k). \tag{65}$$

for all $x_k \in \mathcal{X}_1$.

B. Upper bound Proofs

1) Proof of Lemma 3: On one hand we know that $\sum_{n=N-\tilde{N}+1}^{N-1} \lambda_{h_2}(n|x_k)$ as in (13a) satisfies:

$$\sum_{n=N-\tilde{N}+1}^{N-1} \lambda_{h_2}(n|x_k) \geq V_{\tilde{N}-1}(x_h^*(N-\tilde{N}+1|x_k)), \tag{66}$$

as $V_{\tilde{N}-1}(x_h^*(N-\tilde{N}+1|x_k))$ is optimal. On the other hand:

$$\begin{aligned} V_N^{\tilde{N}}(x_k) \leq \lambda_{h_1}(0|x_k) + \sum_{n=1}^{N-\tilde{N}} \lambda_{h_1}(n|x_k) \\ + V_{\tilde{N}-1}(x_h^*(N-\tilde{N}+1|x_k)), \end{aligned} \tag{67}$$

given $V_N^{\tilde{N}}(x_k)$ is optimal. Using (13a) with (67) produces:

$$\begin{aligned} &\lambda_{h_1}(0|x_k) + \sum_{n=1}^{N-\tilde{N}} \lambda_{h_1}(n|x_k) + \sum_{n=N-\tilde{N}+1}^{N-1} \lambda_{h_2}(n|x_k) = \\ &V_N^{\tilde{N}}(x_k) \leq \lambda_{h_1}(0|x_k) + \sum_{n=1}^{N-\tilde{N}} \lambda_{h_1}(n|x_k) + \\ &V_{\tilde{N}-1}(x_h^*(N-\tilde{N}+1|x_k)). \end{aligned} \tag{68}$$

Canceling out the two first terms to the left of the equality and to the right of the inequality yields:

$$\sum_{n=N-\tilde{N}+1}^{N-1} \lambda_{h_2}(n|x_k) \leq V_{\tilde{N}-1}(x_h^*(N-\tilde{N}+1|x_k)). \tag{69}$$

Together (66) and (69) imply $\sum_{n=N-\tilde{N}+1}^{N-1} \lambda_{h_2}(n|x_k) = V_{\tilde{N}-1}(x_h^*(N-\tilde{N}+1|x_k))$.

2) Proof of Lemma 4: $V_N^{\tilde{N}}(x_{k+1})$ in (7) can be written as

$$V_N^{\tilde{N}}(x_{k+1}) = \sum_{n=0}^{N-\tilde{N}} \lambda_{h_1}(n|x_{k+1}) + \sum_{n=N-\tilde{N}+1}^{N-1} \lambda_{h_2}(n|x_{k+1}). \tag{70}$$

Since $V_N^{\tilde{N}}(x_{k+1}) = V_N^{\tilde{N}}(f(x_k, \mu_N^{\tilde{N}}(x_k))) = V_N^{\tilde{N}}(x_h^*(1|x_k))$, we can upper-bound it by

$$\begin{aligned} &V_N^{\tilde{N}}(f(x_k, \mu_N^{\tilde{N}}(x_k))) \\ &\leq \sum_{n=1}^{N-\tilde{N}} \lambda_{h_1}(n|x_k) + V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}+1|x_k)). \end{aligned} \tag{71}$$

In (71), we have used $\lambda_{h_1}(n|x_k), n = 1, \ldots, N-\tilde{N}$ from (13a) to generate the first $N-\tilde{N}$ terms of $\lambda_{h_1}(n-1|x_{k+1})$ in (70). The remainder of the expression (70) can be constructed by an input taking the system from the state $x_h(N-\tilde{N}|x_{k+1}) = x_h^*(N-\tilde{N}+1|x_k)$ to $x_h(N-\tilde{N}+1|x_{k+1}) \in \mathcal{X}_1$ and any sequence of inputs maintaining subsequent states $[x_h(N-\tilde{N}+2|x_{k+1}), \ldots x_h(N-1|x_{k+1})] \in \mathcal{X}_2$ [7]. Due to Assumption 2, this control sequence always exists and can be obtained by $V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}+1|x_k))$, which is used in the second part of (71). Now, a lower bound for $V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(f(x_k, \mu_N^{\tilde{N}}(x_k)))$ can be obtained by subtracting (71) from (13a), as follows:

$$V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(f(x_k, \mu_N^{\tilde{N}}(x_k)))$$

[7] We have used the dependency on x_{k+1} for predicted states which were not directly obtained using any terms containing x_k as original departure state.

$$\geq \lambda_{h_1}(0|x_k) + \sum_{n=1}^{N-\tilde{N}} \lambda_{h_1}(n|x_k) + \sum_{n=N-\tilde{N}+1}^{N-1} \lambda_{h_2}(n|x_k)$$
$$- \sum_{n=1}^{N-\tilde{N}} \lambda_{h_1}(n|x_k) - V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}+1|x_k))$$
$$= \lambda_{h_1}(0|x_k)$$
$$- \left(V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}+1|x_k)) - V_{\tilde{N}-1}(x_h^*(N-\tilde{N}+1|x_k))\right). \quad (72)$$

Where equality holds using Lemma 3 and cancellation of the summations from $n = 1, \dots, N - \tilde{N}$.

3) Proof of Proposition 2: Analyzing (8) and having an expression for α available, we just need to obtain an upper bound to $V_N^{\tilde{N}}(x_k)$. A closed expression for it can be obtained using (13a). We can then use (23) to bound the first summation in (13a). The second summation is equal to $V_{\tilde{N}-1}(x_s)$ via Lemma 3. Based on this equality we can use (24) and (23) sequentially in $V_{\tilde{N}-1}(x_s)$ to bound the second summation in (13a). This produces

$$V_N^{\tilde{N}}(x_k) \leq \sum_{n=0}^{N-\tilde{N}} C_1\sigma_1^n\lambda_0 + V_{\tilde{N}-1}(x_s)$$
$$\leq C_1\left[\left(\frac{1-\sigma_1^{N-\tilde{N}+1}}{1-\sigma_1}\right) + C_2\sigma_1^{N-\tilde{N}}\sigma_2\left(\frac{1-\sigma_2^{\tilde{N}-1}}{1-\sigma_2}\right)\right]\lambda_0. \quad (73)$$

Using (8) and $\lambda_0 \leq \max_{u\in\mathcal{U}} l(x_k, u)$ produces (29).

4) Proof of Proposition 3: Assume prior knowledge of $C_1, C_2, \sigma_1, \sigma_2$ and δ, then via Theorem 1, we can use N and $\tilde{N}$ to produce $\alpha \geq 0$ as follows.

$$\alpha = 1 - C_1C_2\sigma_1^{N-\tilde{N}}\sigma_2\left(\delta\frac{1-\sigma_2^{\tilde{N}}}{1-\sigma_2} + \sigma_2^{\tilde{N}-1}\right) \geq 0 \quad (74)$$

$$\Rightarrow C_1C_2\sigma_1^{N-\tilde{N}}\sigma_2\left(\delta\frac{1-\sigma_2^{\tilde{N}}}{1-\sigma_2} + \sigma_2^{\tilde{N}-1}\right) \leq 1. \quad (75)$$

To guarantee (75), we upper-bound its left hand-side as

$$C_1C_2\sigma_1^{N-\tilde{N}}\sigma_2\left(\delta\frac{1-\sigma_2^{\tilde{N}}}{1-\sigma_2} + \sigma_2^{\tilde{N}-1}\right)$$
$$< C_1C_2\sigma_1^{N}\left(\delta\frac{\sigma_1(1-\sigma_2^{\tilde{N}})}{\sigma_1^{\tilde{N}}(1-\sigma_2)} + 1\right)$$
$$= C_1C_2\sigma_1^{N-\tilde{N}+1}\left[\frac{\delta}{1-\sigma_2} + 1\right]. \quad (76)$$

The first strict inequality is obtained due to $1 > \sigma_1 > \sigma_2 \geq 0$ in Assumption 3. We then subject

$$C_1C_2\sigma_1^{N-\tilde{N}+1}\left[\frac{\delta}{1-\sigma_2} + 1\right] \leq 1$$
$$\Rightarrow -(N-\tilde{N}+1)\log\left(\frac{1}{\sigma_1}\right) \leq -\log(C_1C_2[\frac{\delta}{1-\sigma 2} + 1])$$
$$\Rightarrow N - \tilde{N} + 1 \geq \left\lceil\frac{\log\left(C_1C_2[\frac{\delta}{1-\sigma_2} + 1]\right)}{\log\left(\frac{1}{\sigma_1}\right)}\right\rceil. \quad (77)$$

5) Proof of Proposition 4: Consider the difference $V_{\tilde{N}}^{\tilde{N}}(x_s) - V_{\tilde{N}}(x_s)$. One could recalculate it as:

$$V_{\tilde{N}}^{\tilde{N}}(x_s) - V_{\tilde{N}}(x_s) = V_{\tilde{N}}^{\tilde{N}}(x_s) - \sum_{n=0}^{\tilde{N}-1} l(x_d^*(n|x_s), u_d^*(n|x_s))$$
$$\leq l(x_h^*(0|x_s), \tilde{u}_h(0|x_s)) + l(x_h(1|x_s), \tilde{u}_h(1|x_s))$$
$$+ \sum_{n=2}^{\tilde{N}-1} l(x_d^*(n|x_s), u_d^*(n|x_s)) - \sum_{n=0}^{\tilde{N}-1} l(x_d^*(n|x_s), u_d^*(n|x_s)). \quad (78)$$

The upper bound on $V_{\tilde{N}}^{\tilde{N}}(x_s)$ is constructed by starting from $x_s = x_h^*(0|x_s) = x_d^*(0|x_s) = x_h^*(N-\tilde{N}+1|x_k)$ and applying any sub optimal input $\tilde{u}_h(0|x_s) \in \mathcal{U}$ producing a state $x_h(1|x_s) \in \mathcal{X}_1$. Then we choose a sub optimal input $\tilde{u}_h(1|x_s)$ that takes the system from $x_h(1|x_s)$ to $x_h(2|x_s) = x_d^*(2|x_s)$. By choosing $\tilde{u}_h(0|x_s)$ and $\tilde{u}_h(1|x_s)$ in this way, we construct a feasible (but generally suboptimal) input sequence, yielding an upper bound on $V_{\tilde{N}}^{\tilde{N}}(x_s)$. This was an arbitrary upperbound sequence choice so that we could cancel out terms $\sum_{n=2}^{\tilde{N}-1} l(x_d^*(n|x_s), u_d^*(n|x_s))$. We then re-write the inequality as:

$$V_{\tilde{N}}^{\tilde{N}}(x_s) - V_{\tilde{N}}(x_s) = V_{\tilde{N}}^{\tilde{N}}(x_s) - \sum_{n=0}^{\tilde{N}-1} l(x_d^*(n|x_s), u_d^*(n|x_s))$$
$$\leq l(x_h(0|x_s), \tilde{u}_h(0|x_s)) - l(x_d^*(0|x_s), u_d^*(0|x_s))$$
$$+ l(x_h(1|x_s), \tilde{u}_h(1|x_s)) - l(x_d^*(1|x_s), u_d^*(1|x_s)). \quad (79)$$

We can further rearrange the above bound to obtain:

$$V_{\tilde{N}}^{\tilde{N}}(x_s) - V_{\tilde{N}}(x_s)$$
$$\leq \left(\frac{l(x_h(0|x_s), \tilde{u}_h(0|x_s))}{l(x_d^*(0|x_s), u_d^*(0|x_s))} - 1\right) l(x_d^*(0|x_s), u_d^*(0|x_s))$$
$$+ \left(\frac{l(x_h(1|x_s), \tilde{u}_h(1|x_s))}{l(x_d^*(1|x_s), u_d^*(1|x_s))} - 1\right) l(x_d^*(1|x_s), u_d^*(1|x_s)). \quad (80)$$

Here, $l(x_d^*(0|x_s), u_d^*(0|x_s)) \neq 0, l(x_d^*(1|x_s), u_d^*(1|x_s)) \neq 0$. If $l(x_d^*(0|x_s), u_d^*(0|x_s)) = 0$, via positive definiteness of $l(\cdot,\cdot)$ with respect to both arguments, this implies $x_d^*(0|x_s) = x_h^*(0|x_s) = 0$, implying $V_{\tilde{N}}^{\tilde{N}}(x_s) = V_{\tilde{N}}(x_s) = 0$, case in which it does not make sense to calculate a bound.If $l(x_d^*(0|x_s), u_d^*(0|x_s)) \neq 0$ and $l(x_d^*(1|x_s), u_d^*(1|x_s)) = 0$, $V_{\tilde{N}}^{\tilde{N}}(x_s) - V_{\tilde{N}}(x_s) \leq l(x_h(0|x_s), \tilde{u}_h(0|x_s)) - l(x_d^*(0|x_s), u_d^*(0|x_s))$ and only the first difference above is to be evaluated.

A particular sub optimal input $\tilde{u}_h(0|x_s)$ choice is to keep the system at x_s or guaranteeing $x_h(1|x_s) = x_h(0|x_s)$. Then $\tilde{u}_h(1|x_s)$ is chosen so that the system goes from $x_h(0|x_s)$ to $x_d^*(2|x_s)$. This scenario is feasible by assumption. As such, we could then simplify the above bound as:

$$V_{\tilde{N}}^{\tilde{N}}(x_s) - V_{\tilde{N}}(x_s)$$
$$\leq \left(\frac{\max_{u\in\mathcal{U}} l(x_h(0|x_s), u)}{\min_{u_1\in\mathcal{U}} l(x_d^*(0|x_s), u_1)} - 1\right) l(x_d^*(0|x_s), u_d^*(0|x_s))$$
$$+ \left(\frac{\max_{u\in\mathcal{U}} l(x_h(0|x_s), u)}{\min_{u_1,u_2\in\mathcal{U}} l(f(x_d^*(0|x_s), u_1), u_2)} - 1\right)$$

$$l(x_d^*(1|x_s), u_d^*(1|x_s))$$

Finally, using Assumption 3, we have:

$$V_{\tilde{N}}^{\tilde{N}}(x_s) - V_{\tilde{N}}(x_s) \leq C_1 C_2 \sigma_1^{N-\tilde{N}} \sigma_2 (\rho_1 + \sigma_2\rho_2)\lambda_0,$$
$$\rho_1 = \frac{\max_{u\in\mathcal{U}} l(x_h(0|x_s), u)}{\min_{u_1\in\mathcal{U}} l(x_d^*(0|x_s), u_1)} - 1 = \frac{\max_{u\in\mathcal{U}} l(x_s, u)}{\min_{u_1\in\mathcal{U}} l(x_s, u_1)} - 1,$$
$$\rho_2 = \frac{\max_{u\in\mathcal{U}} l(x_h(0|x_s), u)}{\min_{u_1,u_2\in\mathcal{U}} l(f(x_d^*(0|x_s), u_1), u_2)} - 1$$
$$= \frac{\max_{u\in\mathcal{U}} l(x_s, u)}{\min_{u_1,u_2\in\mathcal{U}} l(f(x_s, u_1), u_2)} - 1.$$

We assume $\rho_1, \rho_2 \geq 0$. By the non-zero state assumption and the positive definiteness of $l(\cdot,\cdot)$ *in each argument*, the denominators are non-zero. We use this bound to compute α:

$$\alpha = 1 - C_1C_2\sigma_1^{N-\tilde{N}}\sigma_2\left(\rho_1 + \sigma_2\rho_2 + \sigma_2^{\tilde{N}-1}\right)$$
$$\Rightarrow \delta = (\rho_1 + \sigma_2\rho_2)\frac{1-\sigma_2}{1-\sigma_2^{\tilde{N}}}. \quad (81)$$

C. Lower-Bound Proofs

1) Proof of Lemma 5: On one hand we know that $\lambda_{h_1}(N-\tilde{N}|x_{k+1}) + \sum_{n=N-\tilde{N}+1}^{N-1}\lambda_{h_2}(n|x_{k+1})$ satisfies:

$$\lambda_{h_1}(N-\tilde{N}|x_{k+1}) + \sum_{n=N-\tilde{N}+1}^{N-1}\lambda_{h_2}(n|x_{k+1})$$
$$\geq V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}|x_{k+1})), \quad (82)$$

as $V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}|x_{k+1}))$ is optimal. On the other hand, we have:

$$V_N^{\tilde{N}}(x_{k+1}) \leq \sum_{n=0}^{N-\tilde{N}-1}\lambda_{h_1}(n|x_{k+1}) + V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}|x_{k+1})), \quad (83)$$

given $V_N^{\tilde{N}}(x_{k+1})$ is optimal. Using (85) with (83) produces:

$$\sum_{n=0}^{N-\tilde{N}}\lambda_{h_1}(n|x_{k+1}) + \lambda_{h_1}(N-\tilde{N}|x_{k+1})$$
$$+ \sum_{n=N-\tilde{N}+1}^{N-1}\lambda_{h_2}(n|x_{k+1}) = V_N^{\tilde{N}}(x_{k+1})$$
$$\leq \sum_{n=0}^{N-\tilde{N}-1}\lambda_{h_1}(n|x_{k+1}) + V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}|x_{k+1}))$$

Canceling out the first summation term to the left of the equality and to the right of the inequality yields:

$$\lambda_{h_1}(N-\tilde{N}|x_{k+1}) + \sum_{n=N-\tilde{N}+1}^{N-1}\lambda_{h_2}(n|x_{k+1})$$
$$\leq V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}|x_{k+1})). \quad (84)$$

Putting (82) and (84) together implies $\lambda_{h_1}(N-\tilde{N}|x_{k+1}) + \sum_{n=N-\tilde{N}+1}^{N-1}\lambda_{h_2}(n|x_{k+1}) = V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}|x_{k+1}))$.

2) Proof of Lemma 6: We now write $V_N^{\tilde{N}}(f(x_k, \mu_N^{\tilde{N}}(x_k)))$ as:

$$V_N^{\tilde{N}}(f(x_k,\mu_N^{\tilde{N}}(x_k))) = \sum_{n=0}^{N-\tilde{N}-1} l(x_h^*(n|x_{k+1}), u_h^*(n|x_{k+1}))$$
$$+ l(x_h^*(N-\tilde{N}|x_{k+1}), u_h^*(N-\tilde{N}|x_{k+1}))$$
$$+ \sum_{n=N-\tilde{N}+1}^{N-1} l(x_h^*(n|x_{k+1}), u_h^*(n|x_{k+1})) \quad (85)$$

We can get an upper bound on $V_N^{\tilde{N}}(x_k)$ using the first summation in (85) as follows:

$$V_N^{\tilde{N}}(x_k) \leq \lambda_0 + \sum_{n=0}^{N-\tilde{N}-1} l(x_h^*(n|x_{k+1}), u_h^*(n|x_{k+1}))$$
$$+ V_{\tilde{N}-1}(x_h^*(N-\tilde{N}|x_{k+1})). \quad (86)$$

The above is an upper bound as, in the summation, we use a sequence of feasible state input pairs from $V_N^{\tilde{N}}(f(x_k,\mu_N^{\tilde{N}}(x_k)))$ which is not necessarily optimal for $V_N^{\tilde{N}}(x_k)$. The use of $V_{\tilde{N}-1}(x_h^*(N-\tilde{N}|x_{k+1}))$ in $V_N^{\tilde{N}}(x_k)$ can be justified as, from $x_h^*(N-\tilde{N}|x_{k+1})$ onwards, we need a sequence of inputs maintaining the states in $\mathcal{X}_2$. To derive the expression below we use (86) as an upper bound for $V_N^{\tilde{N}}(x_k)$ and the equality (85) for $V_N^{\tilde{N}}(f(x_k,\mu_N^{\tilde{N}}(x_k)))$.

$$V_N^{\tilde{N}}(x_k) - V_N^{\tilde{N}}(f(x_k,\mu_N^{\tilde{N}}(x_k)))$$
$$\leq \lambda_0 + \sum_{n=0}^{N-\tilde{N}-1}\lambda_{h_1}(n|x_{k+1}) + V_{\tilde{N}-1}(x_h^*(N-\tilde{N}|x_{k+1}))$$
$$- \sum_{n=0}^{N-\tilde{N}-1}\lambda_{h_1}(n|x_{k+1}) - \lambda_{h_1}(N-\tilde{N}|x_{k+1})$$
$$- \sum_{n=N-\tilde{N}+1}^{N-1}\lambda_{h_2}(n|x_{k+1})$$
$$= \lambda_{h_1}(0|x_k)$$
$$- \left(V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}|x_{k+1})) - V_{\tilde{N}-1}(x_h^*(N-\tilde{N}|x_{k+1}))\right). \quad (87)$$

Where the equality is obtained via Lemma 5 and cancellation of the summations from $n = 0,\ldots,N-\tilde{N}-1$.

3) Proof of Lemma 7: For any $x \in \mathcal{X}_1$, due to the positive definiteness of $l(\cdot,\cdot)$, we have that

$$V_{\tilde{N}}^{\tilde{N}}(x) \geq V_{\tilde{N}-1}^{\tilde{N}-1}(x) \geq V_{\tilde{N}-1}(x) \implies V_{\tilde{N}}^{\tilde{N}}(x) - V_{\tilde{N}-1}(x) \geq 0,$$

Thus, there exists $\omega \in [0,1)$ (worst case $\omega = 0$) satisfying:

$$(V_{\tilde{N}}^{\tilde{N}}(x) - V_{\tilde{N}-1}(x)) \geq \omega\lambda_0. \quad (88)$$

Multiplying both sides by -1 and adding λ_0 produces:

$$\lambda_0 - (V_{\tilde{N}}^{\tilde{N}}(x) - V_{\tilde{N}-1}(x)) \leq (1-\omega)\lambda_0.$$

Since $1-\omega > 0$, for any non-negative α:

$$\frac{\alpha}{1-\omega}[\lambda_0 - (V_{\tilde{N}}^{\tilde{N}}(x) - V_{\tilde{N}-1}(x))] \leq \alpha\lambda_0. \quad (89)$$

In particular, this will hold for all $\alpha \in [0,1]$ uniformly. Since this holds for any x, it will hold for $x_h^*(N-\tilde{N}|x_{k+1})$. Now consider $x_h^*(N-1|x_{k+1}) \neq 0$. Then, due to the positive definiteness of $l(\cdot,\cdot)$:

$$\begin{aligned}&(V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}|x_{k+1})) > V_{\tilde{N}-1}^{\tilde{N}-1}(x_h^*(N-\tilde{N}|x_{k+1}))\\ &\geq V_{\tilde{N}-1}(x_h^*(N-\tilde{N}|x_{k+1}))).\end{aligned} \tag{90}$$

Following the steps from (88) with the strict inequality $V_{\tilde{N}}^{\tilde{N}}(x_h^*(N-\tilde{N}|x_{k+1})) > V_{\tilde{N}-1}(x_h^*(N-\tilde{N}|x_{k+1}))$ in (90) implies that there will be an $\omega \in (0,1)$ for all $\alpha \in [0,1]$ such that (89) holds. Since the first inequality in (35) always holds, under these conditions Proposition 1 holds.

4) Proof of Proposition 5: Analyzing (12b) and with ω available, we need to obtain an upper bound to $V_N^{\tilde{N}}(x_k)$. We use (41) to bound the first summation. The second one is equal to $V_{\tilde{N}-1}(x_p)$ via Lemma 3, producing:

$$\begin{aligned}&V_N^{\tilde{N}}(x_k) \geq \sum_{n=0}^{N-\tilde{N}} C_3\sigma_3^n\lambda_1 + V_{\tilde{N}-1}(x_p)\\ &\geq C_3\left[\left(\frac{1-\sigma_3^{N-\tilde{N}+1}}{1-\sigma_3}\right) + C_4\sigma_3^{N-\tilde{N}}\sigma_4\left(\frac{1-\sigma_4^{\tilde{N}-1}}{1-\sigma_4}\right)\right]\lambda_0^+.\end{aligned}$$

Where the second inequality is obtained by using (42) and (41) sequentially in $V_{\tilde{N}-1}(x_p)$. Finally, using (12b), $\lambda_0^+ \geq \kappa\lambda_0$ and observing that $\lambda_0 \geq \min_{u\in\mathcal{U}} l(x_k,u)$ results in (47).

5) Proof of Proposition 6: The goal is to have $0 \leq \omega \leq 1-\alpha$. Consider $\alpha \in [0,1]$. If $\sigma_4 = 0$ or $\kappa = 0$, $\omega = 0$, which by default produces $0 \leq \omega \leq 1-\alpha$. Now, we go to the case where $\kappa > 0$, $\omega > 0$. We thus want to enforce

$$\begin{aligned}&C_1C_2\sigma_1^{N-\tilde{N}}\sigma_2\left(\delta\frac{1-\sigma_2^{\tilde{N}}}{1-\sigma_2} + \sigma_2^{\tilde{N}-1}\right)\\ &\geq C_3C_4\kappa\sigma_3^{N-\tilde{N}}\sigma_4\left(\nu\frac{1-\sigma_4^{\tilde{N}}}{1-\sigma_4} + \sigma_4^{\tilde{N}-1}\right)\end{aligned} \tag{91}$$

Since $\sigma_2 \geq \sigma_4$ the expression to the left of the inequality can be lower-bounded as

$$\begin{aligned}&C_1C_2\sigma_1^{N-\tilde{N}}\sigma_2\left(\delta\frac{1-\sigma_2^{\tilde{N}}}{1-\sigma_2} + \sigma_2^{\tilde{N}-1}\right)\\ &\geq C_1C_2\sigma_1^{N-\tilde{N}}\sigma_4\left(\delta\frac{1-\sigma_2^{\tilde{N}}}{1-\sigma_2} + \sigma_2^{\tilde{N}-1}\right) \quad\Rightarrow\\ &\left(\frac{\sigma_1}{\sigma_3}\right)^{N-\tilde{N}}\left[\delta\frac{1-\sigma_2^{\tilde{N}}}{1-\sigma_2} + \sigma_2^{\tilde{N}-1}\right]\\ &\geq \frac{C_3C_4\kappa}{C_1C_2}\left(\nu\frac{1-\sigma_4^{\tilde{N}}}{1-\sigma_4} + \sigma_4^{\tilde{N}-1}\right)\end{aligned} \tag{92}$$

Using the assumptions $\delta \geq \nu$ and again $\sigma_2 \geq \sigma_4$, if we enforce

$$\left(\frac{\sigma_1}{\sigma_3}\right)^{N-\tilde{N}} \geq \frac{C_3C_4\kappa}{C_1C_2} \Rightarrow N-\tilde{N} \geq \left\lceil\frac{\log\left(\frac{C_3C_4\kappa}{C_1C_2}\right)}{\log\left(\frac{\sigma_1}{\sigma_3}\right)}\right\rceil, \tag{93}$$

we produce the desired effect in ω. The case in which $\delta < \nu$ can be analyzed in the same fashion but produces an implicit expression on N and $\tilde{N}$.

6) Proof of Proposition 7: Consider the difference $V_{\tilde{N}}^{\tilde{N}}(x_p) - V_{\tilde{N}}(x_p)$. One could recalculate this difference as:

$$\begin{aligned}&V_{\tilde{N}}^{\tilde{N}}(x_p) - V_{\tilde{N}}(x_p) = \sum_{n=0}^{\tilde{N}-1} l(x_h^*(n|x_p),u_h^*(n|x_p)) - V_{\tilde{N}}(x_p)\\ &\geq \sum_{n=0}^{\tilde{N}-1} l(x_h^*(n|x_p),u_h^*(n|x_p)) - l(x_d(0|x_p),\tilde{u}_d(0|x_p))\\ &- l(x_d(1|x_p),\tilde{u}_d(1|x_p)) - \sum_{n=2}^{\tilde{N}-1} l(x_h^*(n|x_p),u_h^*(n|x_p)).\end{aligned}$$

The upper bound on $V_{\tilde{N}}(x_p)$ is constructed by starting from x_p and applying any sub optimal input $\tilde{u}_d(0|x_p) \in \mathcal{U}$ keeping $x_d(1|x_p) \in \mathcal{X}_2$. Then we choose a sub optimal input $\tilde{u}_d(1|x_p)$ that takes the system from $x_d(1|x_p) = f(x_p,\tilde{u}_d(0|x_p))$ to $x_d(2|x_p) = x_h^*(2|x_p)$. Choosing $\tilde{u}_d(0|x_p)$ and $\tilde{u}_d(1|x_p)$ in this way, a feasible (but suboptimal) input sequence is constructed, yielding an upper bound on $V_{\tilde{N}}(x_p)$. This arbitrary upper bound sequence was chosen to cancel out terms $\sum_{n=2}^{\tilde{N}-1} l(x_h^*(n|x_p),u_h^*(n|x_p))$. Re-writing the inequality as:

$$\begin{aligned}&V_{\tilde{N}}^{\tilde{N}}(x_p) - V_{\tilde{N}}(x_p) = \sum_{n=0}^{\tilde{N}-1} l(x_h^*(n|x_p),u_h^*(n|x_p)) - V_{\tilde{N}}(x_p)\\ &\geq l(x_h^*(0|x_p),u_h^*(0|x_p)) - l(x_d(0|x_p),\tilde{u}_d(0|x_p))\\ &+ l(x_h^*(1|x_p),u_h^*(1|x_p)) - l(x_d(1|x_p),\tilde{u}_d(1|x_p)).,\end{aligned}$$

we can rearrange it to obtain the difference:

$$\begin{aligned}&V_{\tilde{N}}^{\tilde{N}}(x_p) - V_{\tilde{N}}(x_p)\\ &\geq \left(1 - \frac{l(x_d(0|x_p),\tilde{u}_d(0|x_p))}{l(x_h^*(0|x_p),u_h^*(0|x_p))}\right) l(x_h^*(0|x_p),u_h^*(0|x_p))\\ &+ \left(1 - \frac{l(x_d(1|x_p),\tilde{u}_d(1|x_p))}{l(x_h^*(1|x_p),u_h^*(1|x_p))}\right) l(x_h^*(1|x_p),u_h^*(1|x_p))\end{aligned}$$

Following the same reasoning as in the proof of Proposition 4, a feasible input $\tilde{u}_d(0|x_p)$ can be chosen by keeping the system at x_p or guaranteeing $x_d(1|x_p) = x_d(0|x_p)$. Then $\tilde{u}_d(1|x_p)$ would have to make the system go from $x_d(0|x_p)$ to $x_h^*(2|x_p)$, which is feasible by assumption. As such, we could then simplify the above bound as:

$$\begin{aligned}&V_{\tilde{N}}^{\tilde{N}}(x_p) - V_{\tilde{N}}(x_p)\\ &\geq \left(1 - \frac{\max_{u\in\mathcal{U}} l(x_d(0|x_p),u)}{\min_{u_1\in\mathcal{U}} l(x_h^*(0|x_p),u_1)}\right) l(x_h^*(0|x_p),u_h^*(0|x_p))\\ &+ \left(1 - \frac{\max_{u\in\mathcal{U}} l(x_d(0|x_p),u)}{\min_{u_1,u_2\in\mathcal{U}} l(f(x_h^*(0|x_p),u_1),u_2)}\right)\\ &l(x_h^*(1|x_p),u_h^*(1|x_p))\end{aligned}$$

We further use (40) to see that $l(x_h^*(1|x_p),u_h^*(1|x_p)) \geq (1+\nu)l(x_d^*(1|x_p),u_d^*(1|x_p)) \geq l(x_d^*(1|x_p),u_d^*(1|x_p))$. This extra step is needed here since $l(x_h^*(1|x_p),u_h^*(1|x_p))$ does not exist in (41), where the highest index is $l(x_h^*(0|x_p),u_h^*(0|x_p))$. Finally, using Assumption 5, produces:

$$\begin{aligned}&V_{\tilde{N}}^{\tilde{N}}(x_p) - V_{\tilde{N}}(x_p) \geq C_3\kappa\sigma_3^{N-\tilde{N}}\left(\phi_1 + C_4\sigma_4^2\phi_2\right)\lambda_0,\\ &\phi_1 = 1 - \frac{\max_{u\in\mathcal{U}} l(x_d(0|x_p),u)}{\min_{u_1\in\mathcal{U}} l(x_h^*(0|x_p),u_1)} = 1 - \frac{\max_{u\in\mathcal{U}} l(x_p,u)}{\min_{u_1\in\mathcal{U}} l(x_p,u_1)},\end{aligned}$$

$$\phi_2 = 1 - \frac{\max_{u\in\mathcal{U}} l(x_d(0|x_p), u)}{\min_{u_1,u_2\in\mathcal{U}} l(f(x_h^*(0|x_p), u_1), u_2)}$$
$$= 1 - \frac{\max_{u\in\mathcal{U}} l(x_p, u)}{\min_{u_1,u_2\in\mathcal{U}} l(f(x_p, u_1), u_2)}.$$

We could then use this new bound to calculate ω:

$$\omega = C_3\kappa\sigma_3^{N-\tilde{N}}\left(\phi_1 + C_4\sigma_4^2\phi_2 + C_4\sigma_4^{\tilde{N}}\right)$$
$$\Rightarrow \nu = \frac{(\phi_1 + C_4\sigma_4^2\phi_2)}{\sigma_4}\frac{1-\sigma_4}{1-\sigma_4^{\tilde{N}}}. \tag{94}$$

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